

DOUBLE SEMIONS IN ARBITRARY DIMENSION

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ABSTRACT. We present a generalization of the double semion topological quantum field theory to higher dimensions, as a theory of $d - 1$ dimensional surfaces in a d dimensional ambient space. We construct a local Hamiltonian which is a sum of commuting projectors and analyze the excitations and the ground state degeneracy. Defining a consistent set of local rules requires the sign structure of the ground state wavefunction to depend not just on the number of disconnected surfaces, but also upon their higher Betti numbers through the *semicharacteristic*. For odd d the theory is related to the toric code by a local unitary transformation, but for even d the dimension of the space of zero energy ground states is in general different from the toric code and for even $d > 2$ it is also in general different from that of the twisted Z_2 Dijkgraaf-Witten model.

1.

The notion of a topological quantum field theory (TQFT) had its initial success as a $2 + 1$ dimensional theory employed by Witten to give the Jones polynomial a coordinate free interpretation. There are a *few* regimes in which TQFTs generalize straightforwardly to higher dimension, $d + 1$, $d > 2$. These include Dijkgraaf-Witten theories [1] or, more generally, theories based on the category of maps to a space Y with only finitely many nontrivial homotopy groups which are, themselves, finite [6]. Another family of examples are generalized toric codes (GTCs): Given $0 < p < d$ there is a $d + 1$ dimensional TQFT whose Hilbert space on a closed (not necessarily oriented) d -manifold X is spanned by the elements of $H_p(X; Z_2)$. See Ref. [2] for the case $p = 1, d = 2$, see Ref. [3] for the case $p = 1, d = 3$, and see Ref. [4] for the case $p = 2, d = 4$.

In dimension $2 + 1$ the toric code is often studied hand-in-hand with a sister theory “double semions” or DS. In this theory the ground state vectors are multi-loops, with a loop fugacity of $\delta = -1$. The purpose of this paper is to explain (only) in the case $p = d - 1$ how the DS theory can be generalized to $d > 2$. We call this the generalized (double) semion theory (GDS); we parenthesize the word

“double” because these theories are not doubles in any sense that we know. We address:

- (1) gapped local Hamiltonian. This Hamiltonian will be a sum of commuting local terms, each of which is a projector.
- (2) The form of ground state wave functions—answering the question: What is the generalization of loop fugacity $\delta = -1$?
- (3) The nature of loop and dual loop operations, which we now call “balloon” and “dual loop.”, and also the nature of excitations.
- (4) To what class of d -manifolds $\{X^d\}$ should we restrict, to arrive at a theory still analogous to the GTCs, insofar as ground space degeneracies and balloon and loop operators are concerned.

The multi-loops of the theory will be defined on a fixed, finite cellulation of some ambient manifold. Thus, the Hilbert space of the theory will be finite dimensional. Hamiltonians where the terms are commuting projectors are often studied as lattice model realizations of a TQFT. Such lattice model realizations are known in $d = 2$ via the Levin-Wen construction[5] and in $d = 3$ via the Walker-Wang construction[7]. Lattice models where the terms in the Hamiltonian commute are mathematically interesting as these properties simplify the mathematical treatment of these theories; further, stability of these Hamiltonians under small perturbations has been proven under fairly general assumptions[8, 9], so that results about topological order in these idealized Hamiltonians can often be shown to hold for more general Hamiltonians.

2. BRIEF REVIEW OF TORIC CODE (TC) AND DOUBLE SEMIONS (DS)

The generalized toric code model takes as input a cellulation \mathcal{C} of a d dimensional manifold X . There is a qubit degree of freedom for each p cell of \mathcal{C} , for some given p which is also input to the theory, with the Hilbert space \mathcal{H} being the tensor product of these Hilbert spaces (generalizations exists to qudit degrees of freedom; we do not consider this here). The Hamiltonian acting on \mathcal{H} is a sum of two terms,

$$(2.1) \quad H = H_+ + H_\square,$$

where

$$(2.2) \quad H_+ = \sum_{p-1 \text{ cells } e} H_e, \quad H_\square = \sum_{p+1 \text{ cells } c} H_c.$$

We define

$$(2.3) \quad H_e = \frac{1 - \prod_{p \text{ cells } f \text{ s.t. } e \in \partial f} Z_f}{2},$$

where Z_f is the Pauli- Z operator on the cell f and ∂f denotes the boundary of f . We define

$$(2.4) \quad H_c = \frac{1 - \prod_{(d-1) \text{ cells } f \in \partial c} X_f}{2},$$

where the operator X_f is the Pauli- X operator acting on the cell f , changing the state of the cell. When we describe product of several Pauli- X operators on some set of cells later, we will sometimes refer to this as NOT on that set.

These terms are all projectors, and they pairwise commute. The ground state degeneracy is given by $2^{H_p(X, Z_2)}$. A basis for the ground state wavefunctions of this theory is to take equal amplitude superpositions of closed p -chains in a given homology class.

The theory is a topological quantum field theory for $0 < p < d$. We now restrict to $d = 2, p = 1$. On a sphere, the ground state wavefunction of the toric code is a sum over equal amplitude superpositions of closed 1-chains. The double semion model on a sphere has a similar ground state wavefunction up to a sign factor: the ground state wavefunction is a sum over closed 1-chains, with an amplitude equal to -1 raised to a power equal to the number of closed loops. To define this amplitude unambiguously, one typically works on a hexagonal lattice (or other lattice with degree at most 3). If we instead consider a theory of loops on a square lattice, a “figure-8” loop configuration is possible for which it is ambiguous whether we have a single loop which crosses itself or two loops which touch at a corner. If the degree is at most 3 then closed 1-chains unambiguously define a disconnected set of closed loops: a spin \uparrow on a 1-cell corresponds to loop being present on that 1-cell and a spin \downarrow corresponds to loop being absent.

There are two distinct choices typically made for the double semion model Hamiltonian on a hexagonal lattice[5]. Both have the same ground state subspace and both have an excitation gap. In both, we again decompose $H = H_+ + H_\square$, where H_+ is the same as the toric code H_+ for $p = 1, d = 2$ and $H_\square = \sum_{2 \text{ cells } c} H_c$ for H_c defined as follows. In the first choice, one sets

$$(2.5) \quad H_c = \frac{1 + i^{n_{bdry}(c)} \prod_{1 \text{ cells } f \in \partial c} X_f}{2},$$

where $n_{bdry}(c)$ is equal to the number of legs leaving the hexagon c which are in the \uparrow state. Thus, $i^{n_{bdry}(c)}$ is equal to the product of the operators $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ around the legs leaving the hexagon. With this choice, the terms H_c commute with the terms H_e and the terms H_c pairwise commute with each other when restricted to the eigenspace of H_+ with vanishing eigenvalue.

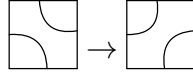
The second choice is to project the term in Eq. (2.5) into the zero eigenspace of the operators H_e for all vertices e in the hexagon c . With this choice, the terms all commute with each other.

Consider a configuration of loops on the sphere. The term H_c can induce a transition between two configurations differing by a surgery move:

$$\smile \rightarrow \rangle \langle$$

To be in the zero eigenspace of H , the two configurations related by this move must enter the wavefunction with the opposite sign. Given any configuration of closed loops on the sphere, indeed this transition will always change the number of loops by ± 1 and hence the sign rule for this transition is consistent with the ground state wavefunction.

However, on other manifolds, such a transition may not change the number of loops. Consider, for example, a single closed loop $(1, 1)$ on a torus, winding once around the meridian and once around the longitude. This can transition to a loop with different relative windings $(1, -1)$, as in the following figure where opposite sides of the square are identified



Thus, the sign structure of the wavefunction on a torus obeys a more complicated rule. In fact, consistency of the transition rule on non-orientable surfaces can reduce the ground state degeneracy of the double semion model; see the next section.

3. SOME PATHOLOGIES OF DS AND ITS POTENTIAL GENERALIZATIONS

For an arbitrary manifold Σ , let $TC(\Sigma)$ and $DS(\Sigma)$ denote the subspace of zero energy states. In this section we consider the degeneracy of this subspace for various manifolds.

Closed non-orientable surfaces are diffeomorphic to: $P, 2P, 3P, \dots$, where P is the projective plane, RP^2 , and the positive integer represents that number of P s tubed (“connected sum”) together.

Theorem 3.1.

$$\dim(DS(tP)) = 2^{t-1}$$

Note. For orientable surfaces, DS and TC have equal dimension, but for closed non-orientable surfaces, $\frac{\dim(DS(tP))}{\dim(TC(tP))} = \frac{1}{2}$.

We will prove theorem 3.1 later in this section. First we collect a few elementary topological facts and then prove some special cases as a warmup before proving the general result.

Fact 3.2. *If a closed surface $\Sigma^2 = \partial M^3$ is the boundary of a 3-manifold,*

$$\dim \ker (H_1(\Sigma, Z_2) \rightarrow H_1(M; Z_2)) = \frac{1}{2} \dim H_1(\Sigma; Z_2).$$

In particular, $\dim H_1(\Sigma; Z_2)$ is even. Equivalently the Euler characteristic $\chi(\Sigma)$ is even.

More generally, if N^{2k} is the boundary of a $(2k+1)$ -dimensional manifold M^{2k+1} , ≥ 1 , then:

$$\dim \ker (H_k(N; Z_2)) \rightarrow H_k(M; Z_2) = \frac{1}{2} \dim H_k(N; Z_2)$$

Again, this implies $\dim H_k(N, Z_2)$ is even, and the Euler characteristic $\chi(N)$ (base of Betti numbers with any coefficients) is also even.

Since we only use Z_2 -coefficients, they are now dropped from the notation.

Proof. Consider the long exact sequences of the pair (M, N) in both cohomology and homology with the vertical isomorphism given by Poincaré and Lefschetz dualities:

$$\begin{array}{ccccccccc} H^k(M, \partial) & \longrightarrow & H^k(M) & \longrightarrow & H^k(\Sigma) & \longrightarrow & H^{k+1}(M, \partial) & \xrightarrow{\hat{\alpha}} & H^{k+1}(M) \\ \downarrow \text{L.D.} & & \downarrow \text{L.D.} & & \downarrow \text{P.D.} & & \downarrow \text{L.D.} & & \downarrow \text{L.D.} \\ H_{k+1}(M) & \xrightarrow{\alpha} & H_{k+1}(M, \partial) & \xrightarrow{\beta} & H_{k+1}(\Sigma) & \xrightarrow{i_*} & H_k(M) & \xrightarrow{\gamma} & H_k(M, \partial) \end{array}$$

The universal coefficient theorem (with field coefficients F) provides a natural isomorphism $H^*(; F) \xrightarrow{\cong} \text{Hom}(H_*(; F), F)$ implying that α and $\hat{\alpha}$ above are Hom-dual. Thus $\text{Hom}(\text{coker } \alpha, Z_2) = \ker \hat{\alpha}$. In particular, $\dim(\text{coker } \alpha) = \dim(\ker \hat{\alpha}) = \dim(\ker \gamma)$, the later from the commutative square. By exactness of the homology sequence, $\text{coker } \alpha \cong \text{image } \beta = \ker i_*$. Thus $\dim(\ker i_*) = \dim(\ker \gamma)$. But using exactness once more, $\text{image } i_* = \ker \gamma$ so we conclude: $\dim(\ker i_*) = \dim(\text{image } i_*)$. Since $\dim H_k(\Sigma) = \dim(\ker i_*) + \dim(\text{image } i_*)$, the fact is proved. \square

Corollary 3.3. *tP , t odd, do not bound any compact 3-manifold. For t even, tP does bound a compact 3-manifold.*

Proof. An elementary calculation shows $H_1(tP) \cong Z_2^t$ and thus has odd dimension for t odd.

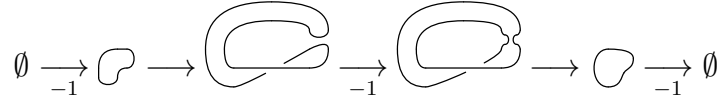
Let $K = 2P$ be the Klein bottle. K is a S^1 -bundle over S^1 with reflection monodromy. (S^1 is the circle.) This monodromy clearly extends to a reflection of the disk D^2 , so K bounds a disk bundle over S^1 . This handles the case $t = 2$. In general, for $t = 2j$ one may tube together j copies of this disk bundle to produce the desired 3-manifold. \square

Fact 3.4. *Any two embedded codimension = 1 submanifolds P and Q of a manifold N^d which represent the same Z_2 -homology class $\alpha \in H_{d-1}(N; Z_2)$ are cobordant: There is a compact W^d with $\partial W^d = P \amalg Q$ (and in fact W also maps into N extending the inclusions of P and Q).*

We will apply this where P and Q are closed surfaces and N is a closed 3-manifold.

Proof. $H_{d-1}(N, Z_2) \cong H^1(N, \partial N, Z_2)$ by Lefschetz duality. H^1 is represented as homotopy classes of maps $[N, RP^\infty]$ (with a relativization on ∂N to the base point if $\partial N \neq \emptyset$). We use ∂X to denote the boundary of any space X . The classifying space for $H^1(\cdot; Z_2)$ is $K(Z_2, 1)$, the space with $\pi_1 \cong Z_2$ and all higher homotopy groups vanishing; it is the infinite projective space RP^∞ . Its characteristic class $\iota \in H^1(RP^\infty; Z_2)$ is dually represented by a codimension one $RP^{\infty-1} \subset RP^\infty$. A class $p \in H^1(N, \partial N; Z_2)$ is represented by a map $f_p : (N, \partial N) \rightarrow (RP^\infty, *)$; $P.D.(p) = [f_p^{-1}(RP^{\infty-1})]$. The submanifolds P and Q arise as different but homotopic ways of making the representing map transverse to $RP^{\infty-1} \subset RP^\infty$. W is produced by a *relative* application of transversality to the homotopy, F . That is, $P = f_p^{-1}(RP^{\infty-1})$, $Q = f_q^{-1}(RP^{\infty-1})$, and $W = F^{-1}(RP^{\infty-1})$. \square

Now consider $DS(RP^2)$. We first consider the question: Is there any ground state wave function (g.s.w.f.) $\psi(\cdot) \in DS(RP^2)$ with nonzero weight on the empty picture or *blank* multi-loop diagram. The answer is no, since starting with the empty picture, a loop may be swept over P with 3 local events, each of which produces a -1 phase difference between the multi-loops before and after each event. The product of these three signs is -1 showing $\psi(\emptyset) = -\psi(\emptyset) = 0$.



In words, we create a small circle in a Möbius band \mathcal{M} , expand it around the band, and recouple it so that it becomes parallel to the boundary of \mathcal{M} . Since $P = \mathcal{M} \cup_\partial \text{disk}$, the recoupled loop now shrinks to a point across the disk and disappears. The sign is $(-1)^{\chi(P)}$ as each “event” corresponds to a critical point of a Morse function on P . In the simplest case there are three. The crossing in the above figure represents the half-twist in the Möbius band; the fourth image (out of six images in the sequence) represents the loop parallel to the boundary of the band. This boundary is drawn, after projecting the band into the plane, as a loop with a single self-crossing.

Similar argument shows that for any DS g.s.w.f. $\psi_{tP}(\text{empty}) = 0$, t odd. The sweepout now has $t + 2$ events: the birth of a circle, t reconnections, and finally the death of the circle. We have shown:

Lemma 3.5. *For t odd, any DS ground state wave function satisfies $\psi_{tP}(\text{empty}) = 0$.*

\square

To gain some intuition, let us now consider a g.s.w.f. $\psi_P(\cdot)$ with $\psi_P(RP^1) = 1$, which evaluates to one on the essential loop. Consider P as the unit disk D

modulo antipodal boundary identifications. There is an embedding of the Klein bottle $K \hookrightarrow P \times S^1$ corresponding to revolving RP^1 (a diameter of D) by π as one traverses the S^1 coordinate.

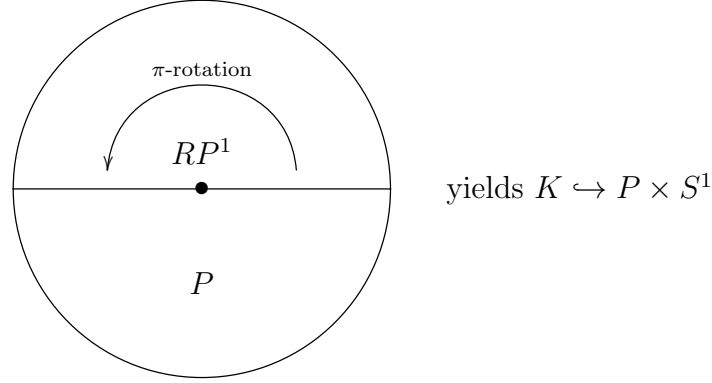


FIGURE 3.1.

Any 1-parameter history of multi-loops including births, deaths, and reconnections beginning and ending with RP^1 defines a closed surface $S \hookrightarrow P \times S^1$. The second homology of $P \times S^1$ obeys the Künneth formula:

$$H_2(P \times S^1) \cong H_1(P) \times H_1(S^1) \oplus H_2(P) \times H_0(S^1) \cong Z_2 \oplus Z_2$$

Choosing the initial multi-loop to be RP^1 or any other choice in that homology class fixes the first factor to be $1 \in Z_2$. The two remaining choices corresponding to $H_2(P) \cong Z_2$ are exemplified by the product history (nothing happens!) where $S \cong S^1 \times S^1 := T$, a torus, and the twisting history (Figure 3.1) where $S \cong K$.

Any other choice of history S' is homologous in $P \times S^1$ to either T or K . Since both T and K are cobordant to \emptyset (bound closed manifolds), and (by Fact 3.4) S' is cobordant to T or K , we see that S' is cobordant to \emptyset . (Concretely, glue the cobordism from S' to T or K to the null cobordism along their common boundary (T or K).) Thus by Fact 3.2, $\chi(S') = \text{even}$. Consequently the skein relations between multi-loops in the non-trivial homology class are consistent: an even number of (-1) s appear as we move from any essential multi-loop configuration through others and finally back to itself. This means that there is a nonzero DS g.s.w.f., with $\psi_P(RP^1) = 1$. With this warmup we can complete the proof of the theorem.

Proof of 3.1. Let Σ be a closed surface with first Z_2 -Betti number $= b$. Let the elements e_1, \dots, e_{2^b} of $H_1(\Sigma)$ span \mathbb{C}^{2^b} . There is a map $\theta, \theta(\psi) = (\psi(E_1), \dots, \psi(E_{2^b}))$ which injects the Hilbert space $\text{DS}(\Sigma)$ into \mathbb{C}^{2^b} , where E_i is any multi-loop representing e_i .

To determine $\dim(\text{DS}(\Sigma))$, it is sufficient to characterize the e_i for which $\psi(E_i)$ can be nonzero.

Claim 3.6. *There is a DS g.s.w.f. ψ_i so that $\psi_i(E_i) = 1$ and $\psi_i(E_j) = 0$, $i \neq j$, if and only if $w_1(e_i) + \chi(\Sigma) = 0$. w_1 is the first Stiefel-Whitney class. $\{\theta(\psi_i)\}$ spans $\text{image } \theta$.*

If Σ is orientable, note that $w_1(e_i) + \chi(S) \equiv 0 + 0 = 0 \pmod{2}$.

Also note the claim implies the theorem since, for a non-orientable $\Sigma = tP$, $w_1 : H_1(\Sigma) \rightarrow Z_2$ is onto, so exactly $\frac{1}{2}$ of the 2^t first homology classes have $w_1(e_i) = 0$ and exactly $\frac{1}{2}$ have $w_1(e_i) = 1$.

Proof of 3.6. As in our warmup discussion of RP^2 , there are precisely two homology classes $E_i \times S^1$ in $\Sigma \times S^1$ which realize a given e_i upon intersection with $\Sigma \times 1$. One is represented by the constant history. Any surface S (history) in this class is cobordant to $E_i \times S^1$ which bounds $E_i \times D^2$, so S is cobordant to \emptyset , hence $\chi(S) \equiv 0 \pmod{2}$. In this case, the skein relations merely say that $\psi(E_i) = (-1)^{\text{even}} \psi(E_i)$ and pose no restriction. The other possibility is more interesting: Σ homologous to $E_i \times S^1 + \Sigma \times 1$. In this case we compute:

$$(*) \quad \chi(S) \equiv w_1(E_i) + \chi(\Sigma) \pmod{2}.$$

$\psi(E_i)$ is forced to vanish exactly when the right-hand side is odd. Thus it suffices to establish (*) for any representative of the cobordism class of S . There is no loss of generality taking E_i to be a single circle, which we do. The natural choice is a “resolution” of the union $(E_i \times S^1) \cup (\Sigma \times 1)$, a union of a torus and a copy of S along $E_i \times 1$. This means that in each normal disk cross-section of $E_i \times 1$ we resolve the crossing $\begin{smallmatrix} \diagup \\ \diagdown \end{smallmatrix}$ of the two surfaces as either $\begin{smallmatrix} \diagup \\ \diagup \end{smallmatrix}$ or $\begin{smallmatrix} \diagdown \\ \diagdown \end{smallmatrix}$. We try to do this continuously all the way around $E_i \times 1$, and we will succeed precisely when $w_1(e_i) = 0$, for in this case $E_i \subset \Sigma$ has a cylindrical neighborhood (as does $E_i \times 1 \subset E_i \times S^1$). We will fail precisely when $w_1(E_i) = 1$, i.e., E_i has a Möbius band neighborhood in Σ . In this case, near the contradictory point we see the two resolutions fitting together to form a “baseball” curve near the final point (see Figure 3.2).

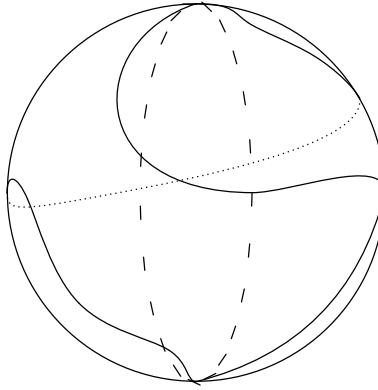


FIGURE 3.2.

Thus, to compute the resolution, a disk δ must be added bounding the baseball curve in the ball in Figure 3.2. Using the additivity formula for Euler characteristic

$$(**) \quad \chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B),$$

one readily checks (*) by showing that when $w_1(E_i) = 0$, $\chi(\text{Res}) = \chi(\Sigma)$ and when $w_1(E_i) = 1$, the partial gluing along two arcs (not circles) contributes -2 to χ while δ contributes $+1$, so in total $\chi(\text{Res}) = \chi(\Sigma) - 1$. \square

\square

Note. As an example, on K any DS g.s.w.f. ψ evaluates to zero on the two non-orientable 1-cycles, but is arbitrary on the oriented cycles. In general the restriction on a DS g.s.w.f. on tP is that it evaluate to zero on non-orientable cycles if and only if t is even and that it evaluate to zero on orientable cycles if t is odd.

In subsequent sections we will identify classes of d -manifolds X for which GDSs on X have ground state degeneracy $(\text{gsd}) = |H_1(X; \mathbb{Z}_2)|$. In the remainder of this section we show a dramatic drop in ground state degeneracy if DS is generalized to a theory of multi-loops of dimension one fluctuating within closed 3-manifolds. This motivates our focus on fluctuating codimension one surfaces within d -dimensional manifolds.

First consider a theory in which *unframed* multi-loops fluctuate in a 3-manifold M^3 with the zero mode ground state wave function (zgswf) experiencing a phase $= -1$ with each Morse transition. Even locally, within a 3-ball, a closed cycle with one Morse transition exists:

$$(3.1) \quad \text{circle} \xrightarrow{\text{isotopy}} \text{figure-eight} \xrightarrow{\text{Morse}} \text{figure-eight} \xrightarrow{\text{isotopy}} \text{circle}$$

The local inconsistency: $\psi(\text{circle}) = -\psi(\text{circle})$ forces any zgswf to be identically zero on any M^3 .

The explanation for this local inconsistency is that we neglected a normal framing, which we now add. So we now assume all Morse transitions respect framings. The beginning and end pictures on line (3.1) now have different framings so there is no inconsistency.

However, even in the context of framed multi-loops, we find that the Hilbert space $V(M)$ of this theory for any closed oriented 3-manifold M has dimension $= 1$, and below (Theorem 3.7) characterize the single nontrivial sector.

Note. It is proved in [7] that for any modular tensor category (MTC), the skein space on a closed oriented 3-manifold has dimension one, so the dimension count in Theorem 3.7 is merely a special case, since “semions” form a MTC.

Given a closed 3-manifold, let $E \in H_1(M) \xrightarrow{\text{P.D.}} H^2(M) \cong \text{hom}(H_2(M); \mathbb{Z}_2)$ denote the class which assigns to any closed surface embedded in M , $\Sigma \hookrightarrow M$, its Euler class $\chi(\Sigma) \pmod{2}$.

Theorem 3.7. *Let M be a closed oriented 3-manifold and let $V(M)$ be the “Hilbert space” of framed multi-loops embedded in M with skein relation: “Morse = -1 ”. $\dim V(M) = 1$, with E being the nontrivial sector.*

Proof. The argument closely parallels the proof of Theorem 3.1, so it is only sketched. Let L be a loop or multi-loop in class $F \in H_1(M)$. Consider a “history” h beginning at $L \times 0 \subset M \times 0 \subset M \times [0, 1]$ and ending at $L \times 1 \subset M \times 1 \subset M \times [0, 1]$ and then identify boundaries to embed $M \times (0, 1) \hookrightarrow M \times [0, 1]/m \times 0 = m \times 1 = M \times S^1$. Let us compute the mod 2 Euler class of the closed history $\bar{h} \subset M \times S^1$. $[\bar{h}] = [L] \times [S^1] + x \times [\text{pt.}] \in H_2(M \times S^1)$, where $x \in H_2(M)$. Using the data of the normal framing, one computes:

$$\begin{aligned} \chi(\bar{h}) &\equiv \chi(L \times S^1) + \chi(x) + |L \times S^1 \cap x| \pmod{2} \\ &\equiv 0 + \chi(x) + F \cdot x \pmod{2} \end{aligned}$$

The right-hand side is zero for all histories, i.e., for all $x \in H_2(M)$, if and only if $F = [L] = E$, which is precisely the case that the sector survives in $V(M)$. \square

4. THE THEORIES WE STUDY: “FLUCTUATING $(d-1)$ SUBMANIFOLD OF X^d ”

A smooth closed X^d manifold will be the home or “ambient space” of our generalized double semion (GDS) theory. We consider a fixed combinatorial structure \mathcal{C} , a generic cellulation, on X . Associated to \mathcal{C} is a “crude Hilbert space” \mathcal{H} consisting of one qubit for each $(d-1)$ -cell of \mathcal{C} , spanned by $\langle \uparrow = \text{present}, \downarrow = \text{absent} \rangle$. There is a Hamiltonian:

$$H = H_+ + H_\square$$

acting on \mathcal{H} . We further write:

$$H_+ = \sum_{d-2 \text{ cells } e} H_e, \quad H_\square = \sum_{d\text{-cells } c} H_c.$$

H_+ has a term for each $(d-2)$ -cell P which is zero if an even number of $(d-1)$ -cells meeting P are \uparrow (present) and 1 if an odd number are \uparrow . This is the same as the H_+ term for the generalized toric code.

H_\square has a term H_c for each d -cell c . This term is

$$(4.1) \quad H_c = \frac{1 - O_c}{2},$$

where

$$(4.2) \quad O_c = \pm \prod_{(d-1) \text{ cells } f \in \partial c} X_f.$$

The operator X_f is the Pauli- X operator acting on the cell f , changing the state of the cell. The sign in (4.2) is $-1 \times -1^{\chi(\uparrow_c)}$ where χ is Euler characteristic and \uparrow_c is the codimension = 0 submanifold of ∂c consisting of the union of $(d-1)$ cells of ∂c which are labeled \uparrow in the state on which H_c acts. We will show in the appendix that this subset is indeed a smooth submanifold with corners. This is the reason that we chose a generic cellulation; it is analogous to the reason for defining the double semion model in two dimensions on a hexagonal lattice, rather than a square lattice.

The boundary of \uparrow_c regarded as a submanifold of ∂c is the same as the boundary of \downarrow_c and is equal to $\uparrow_c \cap \downarrow_c$. For even d , the Euler characteristic of $\uparrow_c \cap \downarrow_c$ is equal to twice the Euler characteristic of \uparrow_c , so $-1^{\chi(\uparrow_c)}$ is equal to $i^{\chi(\uparrow_c \cap \downarrow_c)}$. This makes the sign more reminiscent of the sign in the double semion model in two dimensions, where there is a factor of i for every leg leaving the hexagon.

Explanation of generic cellulations. Generic cellulations may be defined as those divisions of a smooth closed d -manifold X^d into a union of smooth k -cells, $0 \leq k \leq d$, piecewise smooth on their boundaries, which obey the local combinatorics of the dual cells to a smooth triangulation of X . For example, on a surface ($d = 2$), a trivalent graph with contractible complementary regions determines a generic cellulation. In 3D the point singularities are cone over the 1-skeleton of a tetrahedron, etc. A second construction which works in any dimension is to start with not a smooth triangulation but rather Riemannian metric. Then any sufficiently dense set S of points, if perturbed to be generic, will determine Voronoi cells (the d -cell interiors consist of all points closest to some $s \in S$) which define a generic cellulation. We work with these structures rather than some fixed lattice for two reasons:

- (1) every union U of $(d-1)$ cells whose Z_2 -boundary vanishes (\Leftrightarrow every $(d-2)$ cell meets an even number of $(d-1)$ cells of U) is a submanifold.
- (2) heritability: The intersection pattern induced on the boundary of any cell of a generic cellulation is *also* a generic cellulation in a lower dimension. For more details, see appendix.

Lemma 4.1. *All terms of H_+ pairwise commute. All terms in H_+ commute with all terms in H_\square . All terms in H_\square pairwise commute when restricted to the eigenspace of H_+ with vanishing eigenvalue.*

Proof. The terms in H_+ pairwise commute since they are all diagonal in the product basis $|\uparrow\rangle, |\downarrow\rangle$. We assume the reader is familiar with the analogous verification that the terms of Kitaev's [2] toric code (TC) commute. As with TC, adjacent $+$ and \square operators share two anticommuting factors and therefore commute. The local sign is irrelevant.

Consider a pair of terms H_{c_1}, H_{c_2} in H_\square . They trivially commute unless ∂c_1 and ∂c_2 share some $(d-1)$ cell f . Consider a state Ψ in the product basis on which H_+

vanishes. We show that $O_{c_1}O_{c_2}\Psi = O_{c_2}O_{c_1}\Psi$. The states $O_{c_1}O_{c_2}\Psi$ and $O_{c_2}O_{c_1}\Psi$ are both product states and have the same spin configuration. However, we must check that they have the same sign. Suppose first that f is labeled \downarrow in Ψ , so that O_{c_2} changes the set of \uparrow spins on c_1 to $\uparrow_{c_1} \cup f$. So, the sign for $O_{c_1}O_{c_2}$ is equal to $-1^{\chi(\uparrow_{c_1} \cup f)} - 1^{\chi(\uparrow_{c_2})}$. A similar computation for $O_{c_2}O_{c_1}$ gives $-1^{\chi(\uparrow_{c_2} \cup f)} - 1^{\chi(\uparrow_{c_1})}$. So we must check that

$$(4.3) \quad -1^{\chi(\uparrow_{c_1} \cup f)} - 1^{\chi(\uparrow_{c_2})} = -1^{\chi(\uparrow_{c_2} \cup f)} - 1^{\chi(\uparrow_{c_1})}.$$

By additivity formula of Euler characteristic, $\chi(\uparrow_{c_1} \cup f) = \chi(\uparrow_{c_1}) + \chi(f) - \chi(\uparrow_{c_1} \cap f)$. So, we must check that

$$(4.4) \quad \chi(\uparrow_{c_1} \cap f) + \chi(\uparrow_{c_2} \cap f) = 0 \pmod{2}.$$

However, by assumption that H_+ vanishes on Ψ , $\uparrow_{c_1} \cap f = \uparrow_{c_2} \cap f$. If instead f is labeled \uparrow in ψ , we must check that $-1^{\chi(\uparrow_{c_1} \setminus f)} - 1^{\chi(\uparrow_{c_2})} = -1^{\chi(\uparrow_{c_2} \setminus f)} - 1^{\chi(\uparrow_{c_1})}$. A similar calculation using additivity formula for Euler characteristic shows this case also. \square

Lemma 4.2. *All terms of H are projectors.*

Proof. This is obvious for terms of H_+ . For H_c one needs to check that the operator O_c is order two. To do this, one needs to check that the signs agree:

$$(4.5) \quad (-1)^{\chi \uparrow_c} \equiv (-1)^{\chi \downarrow_c} \pmod{2}$$

where \downarrow_c is the complement in ∂c of \uparrow_c . But this follows immediately from three facts:

- (1) ∂c is a $(d-1)$ -sphere S^{d-1} , $\chi(\partial c) = \begin{cases} 0 & d \text{ even} \\ 2 & d \text{ odd} \end{cases}$,
- (2) the additivity formula:

$$\chi(\uparrow_c) + \chi(\downarrow_c) - \chi(\uparrow_c \cap \downarrow_c) = \chi(\partial c) \equiv 0 \pmod{2},$$

and

- (3) Since $(\uparrow_c \cap \downarrow_c)$ bounds (it in fact bounds both \uparrow_c and \downarrow_c), $\chi(\uparrow_c \cap \downarrow_c) \equiv 0 \pmod{2}$ since the Euler characteristic of any bounding manifold is even.

The upshot is (4.3). \square

As in the double semion case in two dimensions, it is possible to define a Hamiltonian which is a sum of commuting projectors by projecting the operators H_c into the zero eigenspace of the terms H_e for e a face of c , setting $H_{\square} = \sum_c H_c^{proj}$, with

$$(4.6) \quad H_c^{proj} = \left(\prod_{d-2 \text{ cells } e, \text{ s.t. } e \text{ is a face of } c} 1 - H_e \right) H_c \left(\prod_{d-2 \text{ cells } e, \text{ s.t. } e \text{ is a face of } c} 1 - H_e \right).$$

Then,

Lemma 4.3. *All terms H_c^{proj} pairwise commute and commute with all terms H_e .*

Proof. The proof is the same as lemma 4.1, noting that that proof only used the vanishing of H_e for H_e a face of c_1, c_2 . \square

Explanation of the sign of H_c . We should think of H_c as inducing fluctuations between codimension one submanifolds. The most basic possible fluctuations are Morse transitions which near the transition point, by definition assume the following local form, for index $0 \leq k \leq d-1$:

$$\begin{aligned} -x_1^2 - x_2^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_{d-1}^2 &= -\epsilon \Rightarrow \\ -x_1^2 - x_2^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_{d-1}^2 &= \epsilon, \epsilon > 0. \end{aligned}$$

For example, for the usual double semions these are the transitions:

$$\begin{aligned} \emptyset &\rightarrow \bigcirc \quad (k=0) \\ \smile &\rightarrow \rangle \langle \quad (k=1) \\ \bigcirc &\rightarrow \emptyset \quad (k=2) \end{aligned}$$

Form the *usual* Skein relations, each of which is associated with a phase $= -1$. We would like to generalize this to higher dimensions by declaring a phase $= -1$ for each Morse transition.

Unfortunately, the transition $t := \uparrow_c \rightarrow \downarrow_c$ is not in general Morse but in general much more complicated. However, a simple argument (below) enables us to write t as a composition of $(\chi(\uparrow_c) + 1)_{\text{mod } 2}$ Morse transitions. This decomposition of t is not unique but the parity of the number of Morse factors is.

Consider a Morse function: $f : c \rightarrow [0, 1]$ with $f^{-1}(0) = (\uparrow_c)^-$, $f^{-1}(1) = (\downarrow_c)^-$, the $(-)$ indicating deletion of a thin collar. Let f be free of critical points on $\partial c \setminus ((\uparrow_c)^- \cup (\downarrow_c)^-)$.

As we cross a critical level s of f , $\chi(f^{-1}(0, s_+])$ changes by ± 1 according to $\text{index}(s) = \text{odd}$ or even , respectively. Since $\chi(c) = \chi(f^{-1}[0, 1]) = 1$, the number (mod 2) of Morse transitions of f is $\chi(\uparrow_c) + 1 = \chi(f^{-1}(0)) + 1$.

Observation 4.4 (“Morse $= -1$ ”). *The Hamiltonian H enforces on any zgswf ψ , the skein rule Morse $= -1$. That is, if E and E' are cellular $d-1$ submanifolds of M which differ by k Morse transitions, $\psi(E) = (-1)^k \psi(E')$.*

5. SOME PROPERTIES OF GDS

Let X^d be the ambient manifold of dimension d . As before, our model is regularized by choosing a generic cellulation \mathcal{C} of X . In this section and Section 6, cutting and gluing constructions are used whose motivation is from the continuum. It may be necessary to subdivide \mathcal{C} to carry out these constructions. Thus all statements should be interpreted as holding for some refinement \mathcal{C}' of \mathcal{C} .

The first property we observe regards the form of zero energy ground state wave functions (zgs wfs). Because of the term H_+ , all zgs wfs are superpositions of (cellular) submanifolds (of dimension $d - 1$, and not necessarily oriented). Because H_\square does *not* fluctuate between different classes of $H_{d-1}(X; Z_2)$, all zgs wfs are superposition of zgs wfs each with support confined to a fixed class $x \in H_{d-1}(X, Z_2)$. The next lemma implies that there is at most one zgs wf up to phase for any class $x \in H_{d-1}(X, Z_2)$. However, as we saw when $d = 2$ on non-orientable surfaces, it is not necessarily the case that *each* possible sector $x \in H_{d-1}(X; Z_2)$ actually admits a zgs wf.

Lemma 5.1. *Let α and β be two \mathcal{C} -cellular $(d - 1)$ - Z_2 -cycles belonging to the same class $x \in H_{d-1}(X; Z_2)$. There is a finite set $\{c_1, \dots, c_j\}$ of d -cells of \mathcal{C} so that $\alpha - \beta = \partial(c_1 \cup \dots \cup c_j)$.*

Proof. This is directly from the definition of cellular homology. No multiplicities or orientation need be considered since the coefficients are Z_2 . \square

Corollary 5.2. *Let \mathbb{C}^{2^b} be the Hilbert space spanned by the elements e_1, \dots, e_{2^b} of $H_{d-1}(X)$, and choose representative cellular cycles E_i for e_i , $1 \leq i \leq 2^b$. The map θ , $\theta(\psi) = (\psi(E_1), \dots, \psi(E_{2^b}))$, is an injection of the zero-modes $\text{GDS}(X) \rightarrow \mathbb{C}^{2^b}$.* \square

In Section 3 we discussed “missing sectors” when X is a non-orientable surface. In higher dimensions, there may be “missing sectors,” θ has cokernel, even when X is orientable. An extreme example is $X = CP^2$.

Example. Let the ambient manifold X be CP^2 , complex projective 2-space, $d = \dim(CP^2) = 4$. $H_3(CP^2, Z_2) \cong 0$, so there is only one possible sector of zgs wfs in GDS, the trivial sector. But, in fact, $\text{GDS}(CP^2) \cong 0$. Just like RP^2 , CP^2 has a Morse function with three critical points

$$(Z_1, Z_2, Z_3) \rightarrow \frac{|Z_2|^2 + 2|Z_3|^2}{|Z_1|^2 + |Z_2|^2 + |Z_3|^2}.$$

Thus, as in Section 3, $\psi(\emptyset) = -\psi(\emptyset) = 0$, so $\text{GDS}(CP^2) \cong 0$. Of course H will have *some* least energy state but we do not know that it has topological significance. We also do not know its entanglement properties. If we work with a Hamiltonian that is a sum of commuting projectors using terms H_c^{proj} , then the ground state is an eigenstate of every term separately. Hence, this ground state will be a zero energy eigenstate of the Hamiltonian on some “punctured cellulation”, obtained by removing the terms H_c, H_e which have expectation value $+1$. We now give precise conditions under which θ is an isomorphism:

Theorem 5.3. *If d is odd, θ is an isomorphism, $\text{GDS}(X) \cong \mathbb{C}^{2^b}$. Moreover, in each sector $x \in H_{d-1}(X)$, an explicit zgs wf is given as $\psi_x(E) = i^{\chi(E)}$.*

For d even and $x \in H_{d-1}(X)$, there is an zgswf $\psi_x(E) \neq 0$, $[E] = x \in H_{d-1}(X)$, if and only if $\chi(X) + \epsilon(x) \equiv 0 \pmod{2}$, where $\epsilon(x)$ is defined below.

In general, there is no local formula¹ for the zgswf; however, if X is a Z_2 -homology d -sphere,² the formula for the sign of ψ_x on E is $\pm = (-1)^{s(E)}$, $s(E)$ also defined below.

Proof. Consider d odd. In this parity, the codimension one cellular submanifold E is even dimensional. Morse transitions of even dimensional manifolds always change χ by ± 2 . The transition removes an $S^j \times D^{d-1-j}$ and replaces it with $D^{j+1} \times S^{d-j-2}$.

$$\chi(S^p) = \begin{cases} 0 & p \text{ odd} \\ 2 & p \text{ even} \end{cases}, \text{ so}$$

$$\chi(S^j \times D^{d-1-j}) = \chi(S^j) \equiv_2 \chi(S^{d-j-2}) = \chi(S^{d-j-2} \times D^{j+1}).$$

The claim now follows from the gluing formula for Euler characteristic:

$$(5.1) \quad \chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$$

Thus every Morse transition $M \rightarrow N$ changes $i^{\chi(M)}$ by a factor of -1 (which is consistent with the fluctuation of H_\square).

Now consider $d = \text{even}$. Let E be a cellular hypersurface of X representing x . Let ν be the normal bundle of E in X and $L \subset E$ be a $d-2$ dimensional embedded submanifold Poincaré dual to the first Stiefel-Whitney class $w_1(\nu) \in H^1(E)$, of the tangent bundle to E .

Definition. $\epsilon(x) = \chi(L) \pmod{2}$.

Lemma 5.4. $\epsilon(x)$ is well-defined.

Proof. According to (3.4), any other hypersurface E' representing x is bordant to E via $V \subset X \times I$; $\partial_+ V = E'$, $\partial_- V = E$. Furthermore, denoting the normal bundle to E' (in $X \times 1$) by ν' , for any L' P.D. to $w_1(\nu')$ in E' there is a bordism (it is the P.D. to $w_1(\nu_{V \hookrightarrow X \times I})$) $\bar{L} \subset V$ from L to L' ; $\partial \bar{L} = L \amalg L'$. Thus L while not uniquely defined is well-defined up to bordism.

From Fact 3.2, if a closed manifold L of dimension $2k$ is a boundary, then $b_k(L)$ is even (and $\dim(\ker_k) = \frac{b_k(L)}{2}$). An immediate corollary is that if L^{2k} is a closed manifold which bounds, then $\chi(L) = \text{even}$, and in particular, if L^{2k} is bordant to L'^{2k} , then $\chi(L) \equiv \chi(L') \pmod{2}$. \square

¹Even when $d = 2$, it is only true on the 2-sphere S^2 that the signs zgswf are $(-1)^{\#(\text{components})}$. Recall the example on the torus T^2 , where the curves $(1, 1)$ and $(1, -1)$ are in the same Z_2 -sector, but $\psi((1, 1)) = -\psi((1, -1))$ for any zgswf ψ .

²It is sufficient to assume $H_{d-1}(X) = 0 = H_{\frac{d}{2}}(X)$.

Now $\psi_x(E)$ can be normalized to 1 unless there is some odd length sequence of Morse transitions in X starting with E and returning to E . That is, unless there is a submanifold $\overline{W} \subset X \times S^1$ with $\overline{W} \cap (X \times 0) = E$ and $\chi(\overline{W}) = \text{odd}$.

But the Künneth theorem tells us there are exactly two possible homology classes (and therefore bordism classes) for \overline{W} . They are $x \times [S^1] \in H_{d-1}(X) \otimes H_1(S^1) \subset H_d(X \times S^1)$, $x \times [S^1] + [X] \times * \in H_{d-1}(X) \otimes H_1(S^1) \oplus H_d(X) \otimes H_1(S^1) = H_d(X \times S^1)$. Again, to determine the parity of $\chi(\overline{W})$, it is sufficient to study any representative of each homology class, since all representatives are bordant. In the first case, $E \times S^1$ is a representative with $\chi(E \times S^1) = 0$, so we may dismiss the first case entirely.

The second case is more interesting and fully analogous to the discussion at the close of Section 3. The class $x \times [S^1] + [X] \times *$ is represented by the union $E \times S^1 \cup X \times 0$. This union is not a submanifold but instead is singular along $E \times 0 = E \times S^1 \cap X \times 0$. If the normal structure, a “ \perp ” bundle over $E \times 0$, is trivial (i.e., a product bundle), the singularity can be resolved (replace \perp with \smile) with $\chi(\text{resolution}) = \chi(E \times S^1) + \chi(X \times 0) = \chi(X \times 0)$. While the S^1 -direction (“vertical”) segment of “ \perp ” is trivial, the horizontal segment is reflected along L ; thus the resolution is as previously pictured in Figure 3.2 but now in a parameter family over L . Where previously a single resolution as in Figure 3.2 accounted for an additional 1 mod 2, now, in its parametric form, it contributes $\chi(L)$. Thus $\chi(\overline{W}) \equiv \chi(X) + \epsilon(x)$. The sector x disappears from $\text{GDS}(X)$ precisely when $\chi(\overline{W})$ is odd.

Thus completes the proof of 5.3 except for the statement about Z_2 -homology spheres which is discussed in Remark 5.6. \square

To make further progress, we need the notion of “Kervaire semi-characteristic.”

We assume $d = 2k + 2$ is even. Kervaire defined the semi-characteristic of a closed odd dimensional manifold M^{2k+1}

$$(5.2) \quad s(M) = \sum_{b=0}^k b_i(M),$$

where b_i is the i^{th} Betti number of M . Kervaire considered real Betti numbers but it is more consistent with our approach to take Z_2 -Betti numbers:

$$(5.3) \quad b_i = \dim(H_i(M; Z_2))$$

and consider $s(M) \in Z_2$, the mod 2 reduction only.

The next lemma says that s fluctuates in a manner similar to the sign θ_c of H_c .

Lemma 5.5. *Suppose M^{2k+1} undergoes a Morse transition to N of index $< k$. Then $s(N) = s(M) + 1$. If index $= k$, this formula still holds if either*

- (1) *the vanishing $(k-1)$ -cycle δ at the transition is nontrivial in $H_{k-1}(M; Z_2)$*
- or*

- (2) $[\delta] = 0$ but the bordism W associated to the transition has vanishing intersection pairing $H_{k+1}(W; Z_2) \times H_{k+1}(W; Z_2) \rightarrow Z_2$.

The intersection pairing clearly vanishes on $H_{k+1}(W; Z_2) \times \text{image}(H_{k+1}(M; Z_2))$ so it is sufficient to consider the intersection pairing on the quotient

$$\mathcal{W} := H_{k+1}(W; Z_2) / \text{image } H_{k+1}(M; Z_2) \cong Z_2$$

$$(5.4) \quad \mathcal{W} \times \mathcal{W} \rightarrow Z_2.$$

Since we are working with Z_2 coefficients, the vanishing of the pairing (5.4) is a nontrivial assumption for W of all even dimensions.

Proof of 5.5. From here forward, Z_2 coefficients are implicit, although most assertions and diagrams have analogs over other fields and sometimes rings.

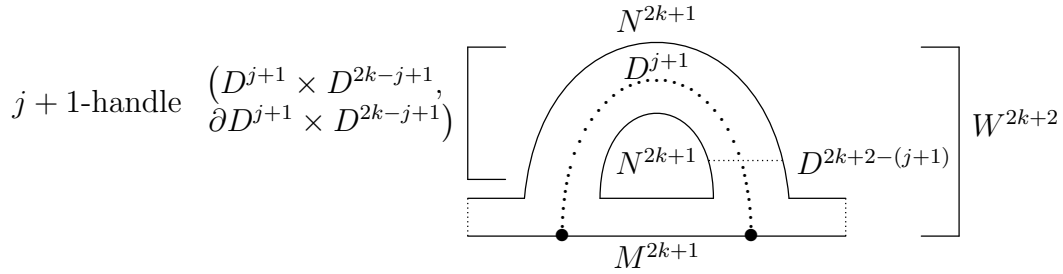


FIGURE 5.1. Curved dotted line D^{j+1} is called “co-core” and horizontal dotted line $D^{2k+2-(j+1)}$ is called “core”. Solid dots indicate $\partial D^{j+1} = S^j$.

The Morse transition of index $j+1$ creates a cobordism W pictured in Figure 5.1. The associate “braided” exact sequences of pairs is displayed below.

$$\begin{array}{ccccccc}
 0 & & & & & & 0 \\
 \parallel & & & & & & \parallel \\
 H_{j+2}(W, M) & \longrightarrow & H_{j+1}(M) & & H_{j+1}(W, N) & \longrightarrow & H_j(N) \\
 & & \searrow & & \nearrow & & \searrow \\
 & & H_{j+1}(W) & & H_j(W) & & \nearrow \\
 & & \nearrow & & \searrow & & \nearrow \\
 H_{j+2}(W, N) & \xrightarrow{\partial_{j+2}} & H_{j+1}(N) & & H_{j+1}(W, M) & \xrightarrow{\partial_{j+1}} & H_j(M) \\
 & & \nearrow & & \searrow & & \nearrow \\
 & & H_j(W, N) & & H_j(W, N) & & \xrightarrow{\partial_j}
 \end{array}$$

\parallel
 Z_2

$$H_*(W, M) \cong \begin{cases} Z_2 & * = j+1 \\ 0 & * \neq j+1 \end{cases} \text{ and } H_*(W, N) \cong \begin{cases} Z_2 & * = 2k-j+1 \\ 0 & * \neq 2k-j+1 \end{cases}$$

First assume $j < k$ making $H_*(W, N) \cong 0$, $* \leq j + 2$. Then $H_j(N) \cong H_j(W)$. Suppose $\partial_{j+1} = 0$, then also $H_j(M) \cong H_j(W)$, so $H_j(N) \cong H_j(M)$ and $H_{j+1}(N) \cong H_{j+1}(W) \cong H_{j+1}(M) \oplus Z_2$. On the other hand, if $\partial_{j+1} \neq 0$ then $H_j(M) \cong H_j(N) \oplus Z_2$ and $H_{j+1}(M) \cong H_{j+1}(N)$. Clearly for $i \leq k$ and $i \leq j$ or $j + 1$, $H_i(M) \cong H_i(W) \cong H_i(N)$. Thus in either case, $\partial_{j+1} = 0$ or $\partial_{j+1} \neq 0$, the semi-characteristic $s \in Z_2$ satisfies $s(N) = s(M) + 1$.

Now consider the more delicate case $j = k$.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_{k+1}(M) & & Z_2 & \xrightarrow{\partial'} & H_k(N) & & 0 \\
 & & \searrow & & \nearrow & & \searrow & & \\
 & & H_{k+1}(W) & & & & H_k(W) & & \\
 & & \nearrow & & \searrow & & \nearrow & & \\
 0 & \longrightarrow & H_{k+1}(N) & & Z_2 & \xrightarrow{\partial} & H_k(M) & & 0
 \end{array}$$

Again consider cases. For ∂, ∂' zero or nonzero, we have the following table for $s(N) - s(M)$:

	$\partial = 0$	$\partial \neq 0$
$\partial' = 0$	0	1
$\partial' \neq 0$	1	*

The * indicates that this case does not actually occur. It is forbidden by Poincaré duality. If $\partial(1) = y \neq 0 \in H_k(M)$, y will be Poincaré dual to some $\hat{y} \in H_{k+1}(M)$; $y \cdot \hat{y} = 1$. But a punctured copy of \hat{y} is a null homology for $\partial'(1)$ as sketched in Figure 5.2.

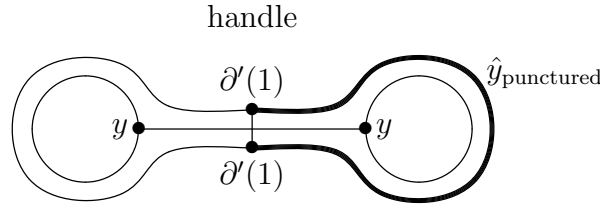


FIGURE 5.2.

The $(0, 0)$ case occurs precisely in the forbidden case where the cobordism W has a $(k+1)$ -cycle with odd self-intersection, or from line (5.4) $\mathcal{W} \times \mathcal{W} \rightarrow Z_2$ is nonzero. One way to compute this is to take the obvious dual disks (called “core” and “co-core”) $D^{k+1} \times 0$ and $0 \times D^{k+1}$ in the $k+1$ -handle $(D^{k+1} \times D^{k+1}, \partial D^{k+1} \times D^{k+1})$ of W relative M and close them off by chains, one in M exhibiting a null homology for $\partial(1)$ and one in N exhibiting a null homology for $\partial'(1)$. This is illustrated in Figure 5.3 where W is drawn as a punctured Möbius band.

□

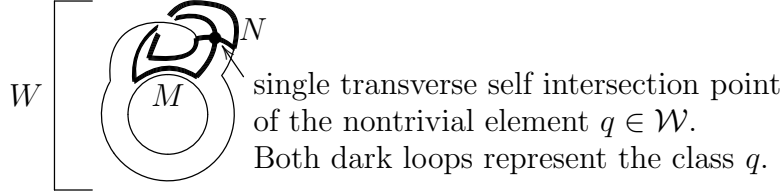


FIGURE 5.3.

Remark 5.6. *This remark completes the proof of Theorem 5.3; the statement about manifolds X with $H_*(X) \cong H_*(S^d)$. When $H_{d-1}(X) \cong 0$, each elementary bordism W across a plaquette (i.e., d -cell) will necessarily be orientable and hence (unlike the bordism in T^2 between $(1, 1)$ and $(1, -1)$) be represented as a codimension zero submanifold of X , $W \subset X$. This and $H_{\frac{d}{2}}(X) \cong 0$ implies that even when Morse transitions are of index k , $d = 2k + 2$, condition (2) of (5.5) will hold, whether or not $[\delta] = 0$. So we conclude that all Morse transitions of $E \subset X$ change the parity of $s(E)$. Furthermore, in the previous part of the proof we established that $\chi(\overline{W}) = \text{even}$ in the present case since $\chi(X) = \chi(S^d) = 2$ and $\epsilon(0) = 0$. These two facts imply that $(-1)^{s(E)}$ is a consistent assignment of signs for a nontrivial (and unique up to scale) zgsuf in $\text{GDS}(X)$. This completes the last portion of Theorem 5.3. Note that the proof only requires $H_{d-1}(X) = 0 = H_{\frac{d}{2}}(X)$, a weaker assumption than assuming X is a \mathbb{Z}_2 -homology sphere.*

Remark 5.7. *For d odd, Theorem 5.3 gives an explicit form of the ground state wavefunction in each sector as $\psi_x(E) = i^{\chi(E)}$. We claim that the unitary U that multiplies each basis vector by a phase $i^{\chi(E)}$ can be written as a local quantum circuit, i.e., with a bounded depth of the circuit and with the gates of the circuit supported on sets of bounded diameter, with these bounds depending only on local geometry. Thus, there is a local quantum circuit that maps the ground state subspace of the GDS for d odd onto the ground state subspace of a GTC. Further, since every Morse transition for odd d changes χ by ± 2 , if we restrict to the ground state subspace of H_+ then the unitary U conjugates the operator H_\square of the GDS to the operator H_\square of the GTC (we leave it as an open problem whether there is a relationship between the theories if we consider the space of states outside the ground state subspace of H_+).*

To construct the local quantum circuit, write

$$(5.5) \quad U = \prod_{j=0}^{d-1} \prod_{j\text{-cells } c} U(j, c),$$

where the second product is over all j -cells c , and $U(j, c)$ is a unitary that multiplies basis vectors by phase equal to either 1 or $\pm i$, with the phase being 1 if the given cell c is not in E and $\pm i$ if c is in E (the phase being $+i$ if j is

even and $-i$ if j is odd). Each $U(j, c)$ is supported only on the $(d - 1)$ cells that intersect the given j -cell c . Those unitaries are the gates in the quantum circuit. Unitaries supported on disjoint sets can be performed in the same round, and in a bounded number of rounds all unitaries can be performed, assuming the cellulation has bounded local geometry, i.e., assuming there is a bound on the number of $(j + 1)$ cells meeting any j cell and also a bound on the number of j -cells in the boundary of any $(j + 1)$ cell and a bound on the diameter of any cell.

6. GENERALIZED LOOP, OR “BALLOON” OPERATORS

Let us recall the Wilson loop “ W_l ” operators [5] for the usual TC and DS theories. In the TC case, one merely specifies a loop l , the support of W_l , and then W_l acts as NOT on the bonds of l . In the case of DS, this definition of W_l is augmented by a sign rule (e.g., line 44 of [5]) which is necessary to preserve ground states. The rule stated in [5] for a loop l or the 2-sphere S^2 may be restated in topological language as the sign rule below.

Let α be the multi-loop of \uparrow bonds. The sign for the action of W_l on $\psi(\alpha)$ is:

$$(6.1) \quad -1^{s(l)} i^{\chi(\partial(l \cap \alpha))} (-1^{\text{link}(\partial(l \cap \alpha))})$$

Again s is \mathbb{Z}_2 -semi-characteristic, which is 1 for a simple closed curve l . ($l \cap \alpha$) consists of line segments (or all of l if $l \subset \alpha$), so $\partial(l \cap \alpha)$ is an even number of points and χ merely counts them. These points are naturally divided into pairs (“0-spheres”) as the boundaries of the component arcs of $\alpha \setminus l$. For a collection of 0-spheres in a circle l , there is a total mod 2 linking number which we denote by “link,” which can be computed, for example, by making l bound a disk Δ , then spanning all the 0-spheres by simple arcs in Δ , and finally counting (mod 2) the number of pairwise intersections of the arcs. An equivalent definition is to reflect all component arcs in $\alpha \setminus l$ which lie outside Δ into the interior of Δ , and then count pairwise intersections. An example is given in Figure 6.1.

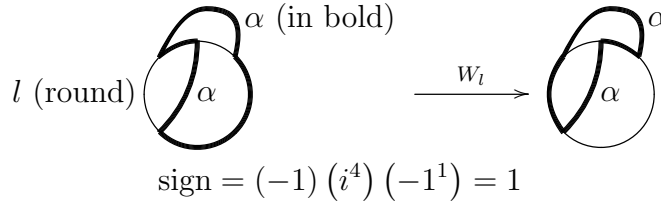


FIGURE 6.1.

Our task in this section is to define the correct signs for the analogous Wilson “balloon” operator. To obtain explicit formulas, we introduce restrictions on the topology of X as necessary. These operators are equal to \pm NOT on an imbedded codimension = 1 \mathcal{C} -cellular submanifold $L^{d-1} \subset X^d$, up to a sign. To be consistent,

the sign \pm must be set so as to carry zgs wfs to zgs wfs. Keeping similar notation, α is the $(d-1)$ -dimensional cellular submanifold supporting a basis vector and L the Wilson balloon.

Note that in addition to these Wilson loop operators W_l , the operator given by the product of Pauli Z around any closed loop on the dual lattice also commutes with the Hamiltonian of both the TC and DS models. This operator (which we will call a dual Wilson loop) straightforwardly generalizes to the GTC and GDS. For the GDS, given any closed loop on the dual lattice, the product of Pauli Z over all $(d-1)$ -cells that intersect that loop is an operator that commutes with the Hamiltonian. If the closed loop is nullhomologous, then this operator acts as the identity on the ground state subspace.

Theorem 6.1. *Let $d = 2k + 2$, $k > 0$. As in 5.3, we require that X^d is a Z_2 -homology sphere.³ Without this assumption, a local description of the balloon operator signs may not be possible. The consistent sign for Wilson balloon operators is:*

$$(-1)^{s(L)} (i^{\chi(\partial(L \cap \alpha))})$$

For the case of odd dimensional d , let $d = 2k + 1$. Then for all closed X^d (no restriction on topology), the consistent sign is:

$$i^{\chi(L)} (-1)^{\chi(L \cap \alpha)}$$

Note. The formula in odd dimensions is quite elementary, involving only Euler characteristic. In even dimensions, the formula is curiously simpler than the classical $d = 2$ case of DS. If one traces the reason back through the proof, the anomaly for $d = 2$ is that $\partial(L \cap \alpha)$ is then zero dimensional, i.e. points. In this very low dimension, the middle dimensional (zero) homology of $S^0 \times S^0$ a product of spheres has rank 4 not 2 and the usual analysis of “hyperbolic pairs,” forms $\begin{vmatrix} 0 & 1 \\ \pm 1 & 0 \end{vmatrix}$ or over Z_2 simply $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$, does not apply.

Proof of 6.1. Consider first d odd. Write $L = A \cup B$, and $\alpha = B \cup C$, $M = \partial A = \partial B = \partial C$, as shown in Figure 6.2, and apply the additivity formula (5.1) for Euler characteristic.

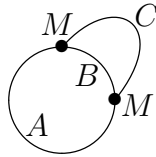


FIGURE 6.2.

³It is sufficient to assume $H_{d-1}(X) \cong 0 \cong H_{\frac{d}{2}}(X)$ and that the Stiefel-Whitney classes w_i of the tangent bundle $\tau(X)$ vanish for $i \leq \frac{d}{2}$.

(6.2)

$$\begin{aligned}\chi(A \cup C) - \chi(B \cup C) &= \chi(A) + \chi(C) - \chi(M) - \chi(B) - \chi(C) + \chi(M) = \\ \chi(A) - \chi(B) &= \chi(A) + \chi(B) - 2\chi(B) = \chi(A \cup B) + \chi(M) - 2\chi(B) = \\ &= \chi(L) + \chi(M) - 2\chi(L \cap \alpha).\end{aligned}$$

Since M is a closed odd dimensional manifold, $\chi(M) = 0$. Thus the change in sign of a configuration in the gswf upon applying W_L is as claimed.

Now consider $d = 2k + 2$, even. To handle this case, we need a rather powerful algebraic lemma 6.3 which, since we are using Z_2 -coefficients, may be stated uniformly regardless of whether k is even or odd. Lemma 6.3 will be applied to the three kernels

$$\ker(H_k(M) \rightarrow H_k(A)), \ker(H_k(M) \rightarrow H_k(B)), \text{ and } \ker(H_k(M) \rightarrow H_k(C))$$

which, in a slight abuse of notation, are also denoted A , B , and C , respectively.

Lemma 6.2. *If the Stiefel-Whitney classes of the tangent bundle $w_i(\tau(X^{2k+2}))$ vanish for $i \leq k = (\frac{d}{2} - 1)$, then the intersection form on $H_k(M^{2k}; Z)$ is isomorphic to a direct sum $Q = \bigoplus_{i=1}^j \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.*

Proof. The total Stiefel-Whitney class of $\tau(M)$, $w = 1 + w_1 + w_2 + \cdots + w_{2k}$, satisfies $w = \text{Sq}(v)$ where $v = 1 + v_1 + v_2 + \cdots + v_{2k}$ is the total Wu class and Sq is the total Steenrod square $\text{Sq} = 1 + \text{Sq}^1 + \text{Sq}^2 + \cdots$. Inductively we find:

$$\begin{aligned}w_1 &= v_1 \\ w_2 &= v_2 + v_1 \cup v_1 \\ w_3 &= v_3 + \text{Bockstein}(v_2) \\ w_4 &= v_4 + \text{Bockstein}(v_3) + v_2 \cup v_2 \\ &\vdots\end{aligned}$$

so $w_i = 0$, $i \leq k$, implies $v_i = 0$, $i \leq k$.

We have assumed that $w_i(\tau(X)) = 0$, $i \leq k$. However, $\tau(X)|_M = \tau(M) \oplus \nu_{M \hookrightarrow X}$ and $\nu_{M \hookrightarrow X}$ is trivialized by the cross-sections (inward normal to A , inward normal to B). Thus, for all i , $w_i \tau(M) = \text{inc}^* w_i(\tau(X))$, implying that $w_i(\tau(M)) = 0$, $i \leq k$. Hence the k^{th} Wu class of M , $v_k = 0 \in H^k(M)$, vanishes.

$\text{Sq}^k(x) = v_k \cup x$ for $x \in H^k(M)$ so:

$$x \cup x = v_k \cup x,$$

so the vanishing of v_k implies that for all $x \in H^k(M)$, $x \cup x = 0$. It is now an elementary exercise in duality to put the intersection form of M in the claimed coordinates. \square

In Lemma 6.3, Z_2 is called F_2 , the field of two elements, to emphasize that it is a field.

Lemma 6.3. *Consider a space F_2^{2j} , with coordinates labelled $1, 2, \dots, 2j$. Let Q be the form*

$$(6.3) \quad Q = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & \dots \end{pmatrix}.$$

Let A, B, C be three maximal subspaces (i.e., j -dimensional subspaces) such that Q vanishes on each subspace. Assume also that for any triple of vectors $a \in A, b \in B, c \in C$ such that $a + b + c = 0$ we have

$$(6.4) \quad (a, Qb) = 0.$$

Then

$$(6.5) \quad \dim(A \cap B) + \dim(B \cap C) + \dim(C \cap A) = j \pmod{2}.$$

Since we will give an inductive proof and we will want to show that this last condition Eq. (6.4) holds under the induction, we will be referring to it below; so, we give it a name, referring to it as the “no twist” condition. Before giving the proof, we remark that the no twist condition is symmetric under interchange of the subspaces A, B, C , because $(b, Qc) = (b, Qa) + (b, Qb) = 0$ and similarly $(a, Qc) = 0$.

Proof. Assume that $\dim(A \cap B) \neq 0$. In this case we will give an inductive proof, reducing the problem to a problem with j reduced by 1. Apply a basis transformation so that A and B both contain the vector $v_1 \equiv (1, 0, 0, 0, \dots)$. Then, every vector in A, B vanishes on the second entry. So, A can be written as the span of v_1 and A' and B can be written as the span of v_1 and B' , where A' and B' are $j - 1$ -dimensional subspaces with Q vanishing on A' and B' and every vector in A' or B' vanishing on the first two entries. Suppose that v_1 is also in C or that $v_2 = (0, 1, 0, 0, \dots)$ is in C . Then, C can also be written as the span of v_1 and C' or v_2 and C' , with C' vanishing on the first two entries. Then, the subspaces A', B', C' are maximal subspaces of a space $F_2^{2(j-1)}$ with Q vanishing on A', B', C' and the no twist condition holding for A', B', C' because A', B', C' are subspaces of A, B, C respectively. Further,

$$(6.6) \quad \begin{aligned} &\dim(A \cap B) + \dim(B \cap C) + \dim(C \cap A) = \\ &\dim(A' \cap B') + \dim(B' \cap C') + \dim(C' \cap A') + 1 \pmod{2}, \end{aligned}$$

as desired.

So, suppose that v_1 and v_2 are not in C . Then, it cannot be the case that every vector in C vanishes on the first coordinate (else v_2 must be in C) or that every

vector in C vanishes on the second coordinate (else v_1 must be in C). So, one of two cases hold. Either C contains a vector $w = (1, 0, w_3, w_4, \dots)$ and a vector $w' = (0, 1, w'_3, w'_4, \dots)$ or C contains a vector $x = (1, 1, x_3, x_4, \dots)$ but does not contain any vector of the form w or w' .

Suppose the first case holds, that C contains such a w and w' . So, C can be written as the span of w, w' and some subspace C'_0 that vanishes on the first and second coordinates. Let y be the vector $(0, 0, w_3, w_4, \dots)$. Define C' to be the span of C'_0 and y . Note that y is not in C'_0 by the assumption that v_1 is not in C so C' has dimension $j - 1$. Note that $\dim(A \cap C) = \dim(A \cap \text{span}(w, C'_0)) = \dim(A' \cap C')$ and similarly $\dim(B' \cap C') = \dim(B \cap C)$. So, this again gives three subspaces A', B', C' for which Eq. (6.6) holds and the no twist condition holds so again we reduce j by 1.

Suppose instead the second case holds, that C contains a vector

$$x = (1, 1, x_3, x_4, \dots)$$

but does not contain any vector of the form w or w' . Define C' to be the projection of C onto the space on which the first two coordinates vanish; i.e., C' is the space of vectors $(0, 0, v_3, v_4, \dots)$ such that $(0, 0, v_3, v_4, \dots)$ is in C or $(1, 1, v_3, v_4, \dots)$ is in C . Being a projection, C' has dimension j or $j - 1$. The form Q vanishes on C' so C' has dimension $j - 1$. So, if a vector $(1, 1, x_3, x_4, \dots)$ is in C then $(0, 0, x_3, x_4, \dots)$ is in C . So, $(1, 1, 0, 0, \dots)$ is in C so C is the span of $(1, 1, 0, 0, \dots)$ and C' , so $\dim(A' \cap C') = \dim(A \cap C)$ and $\dim(B' \cap C') = \dim(B \cap C)$ so Eq. (6.6) holds. Further, the no twist condition holds for A', B', C' .

So, we succeed in reducing j by 1 whenever $\dim(A \cap B) \neq 0$. A similar reduction can be performed whenever $\dim(A \cap C) \neq 0$ or $\dim(B \cap C) \neq 0$ so we can assume that $A \cap B = B \cap C = C \cap A = \emptyset$. We now show that this implies that j is even.

Make a basis transformation leaving Q invariant so that A is the space of vectors vanishing on the even coordinates. Introduce notation, writing a vector $v \in F_2^{2j}$ as $v = (x; y)$ where x, y are j -dimensional vectors to indicate that the odd coordinates of v are the entries of x while the even coordinates are the entries of y . Parametrize B as the space of vectors $(Ku; Lu)$ for two matrices K, L where u is a j -dimensional vector. Since B is j -dimensional and $A \cap B = \emptyset$, L is non-singular. So, we can instead parametrize B as the space of vectors $(KL^{-1}y; y)$ or $(My; y)$ where $M = KL^{-1}$. Similarly, we can parametrize C as the space of vectors $(Ny; y)$. The matrices M, N are symmetric because Q vanishes on B, C . We now use the no twist condition. We have $((M + N)y; 0) + (My; y) + (Ny; y) = 0$, and so $(My, y) + (Ny, y) = 0$ for all y or equivalently $(y, (M + N)y) = 0$. So, $M + N$ vanishes on the diagonal. However, $M + N$ must be nonsingular (otherwise, $B \cap C \neq \emptyset$). So, j must be even.

The fact that j must be even follows from the fact that any symmetric, non-singular matrix over F_2 which vanishes on the diagonal must be even dimensional. To show this, consider any such matrix F . If G is a non-singular matrix, then

$G^T F G$ is still a symmetric non-singular matrix which vanishes on the diagonal. We use two different choices of G , either a permutation matrix or a matrix which is 1 on the diagonal and has one nonzero entry of the diagonal, to simplify the matrix F through a sequence of steps similar to Gaussian elimination; the second choice of G gives a “row and column” operation. F must have a nonzero entry on the first row (otherwise it is singular), so use permutations to make the upper right corner of F nonzero. Then, use “row and column” operations to make all entries in the last column vanish, except for the entry in the upper right corner. Repeat this procedure on the second row, and so on, continuing until all the nonzero entries are on the diagonal from top right to bottom left corner. This diagonal intersects the main diagonal if j is odd (and so the matrix must be singular), so j must be even. \square

Again, the notation of the lemma is designed to evoke Figure 6.2, with a minimal abuse of notation. We wrote A , B , and C for the kernels of the three maps:

$$H_k(M) \rightarrow H_k(A)$$

$$H_k(M) \rightarrow H_k(B)$$

$$H_k(M) \rightarrow H_k(C).$$

As we now show, these kernels indeed have the “no twist” (nt) property:

$$(nt) \quad a + b + c = 0 \ (a \in A, b \in B, c \in C) \text{ implies } (a, Qb) = 0.$$

Lemma 6.4. *Under the assumption $w_i(\tau(X)) = 0$, $i \leq k + 1 = \frac{d}{2}$, (nt) holds.*

Proof. First recall from the proof of Lemma 6.2 that $w_i(\tau(X)) = 0$, $1 \leq i \leq k + 1$, implies that two Wu classes vanish, $v_{k+1} \subset H^{k+1}(X)$ and $v_k \in H^k(M)$. The first vanishing is equivalent to $x \cdot x = 0$ for all $x \in H_{k+1}(X)$ and the second vanishing is equivalent to $y \cdot y = 0$ for all $y \in H_k(M)$.

Assuming $(a, Qb) = 1$, we may build a cycle $x \in H_j(X)$ with $x \cdot x = 1$, a contradiction. The construction is to form V representing x by gluing together three coboundaries: $V = \delta_a^A \cup \delta_b^B \cup \delta_c^C$ where the superscript is the ambient space of the coboundary, and $\partial(\delta_p^- = p)$.

The plan is to compute $x \cdot x$ taking V and a copy \tilde{V} perturbed in X to be in general position with respect to V and counting intersection points. The answer is $x \cdot x = |V \cap \tilde{V}| = (a, Qb)$, but from our assumptions on X , $x \cdot x = 0$, implying $(a, Qb) = 0$.

Although the submanifolds A , B , and C play symmetric roles, it is helpful for the computation to *break symmetry* and consider $a \cup b \subset L := A \cup B$. We may obtain a second copy $a' \cup b'$ of $a \cup b$ by perturbing this k -cycle toward A using the normal direction of $M \subset A$. Let $\delta_a^{A'}$, $\delta_b^{B'}$, and $\delta_c^{C'}$ be formed by extending this

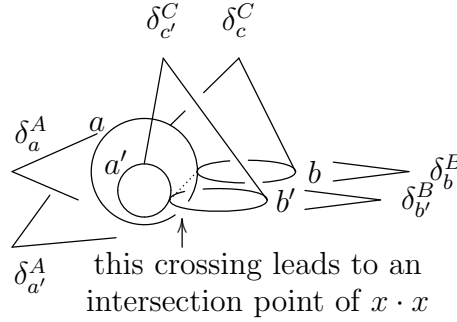


FIGURE 6.3.

perturbation over the three δ s. There is a Z_2 -linking number defined:

$$l(a \cup b, a' \cup b') = (a \cup b) \cdot (\delta_a^{A'} \cup \delta_b^{B'}) = (\delta_a^A \cup \delta_b^B) \cdot (a' \cup b').$$

As in the construction of the classical Seifert form in knot theory,

$$l(a \cup b, a' \cup b') = l(a, a') + l(a, b') + l(b, a') + l(b, b') :=$$

$$a \cdot \delta_a^{A'} + a \cdot \delta_b^{B'} + b \cdot \delta_a^{A'} + b \cdot \delta_b^{B'} = 0 + [a] \cdot [b] + [b] \cdot [a] = 0 + (a, Qb) + 0$$

where the first term is zero since the perturbation is towards A , the $[\]$ denote classes in $H_k(M)$, and the “dot” on the last line denotes intersection in M . The last term is zero since $V_k = 0 \in H^k(M)$.

Let $V' = \delta_a^{A'} \cup \delta_b^{B'} \cup \delta_c^{C'}$. $V \cap V'$ consists of a collection of arcs counted by $a \cdot b \equiv_2 (a, Qb)$. Now form \tilde{V} by further perturbing V' to be transverse to V in X . Each arc of $V \cap V'$ becomes a transverse intersection point of $V \cap \tilde{V}$. Thus we conclude $x \cdot x \equiv_2 |V \cap \tilde{V}| \equiv_2 (a, Qb)$, provided all *other* points of $V \cap \tilde{V}$ come in *pairs*.

Before we perturbed V' to \tilde{V} , $V \cap V'$ consists of the just described arcs (counted by $a \cdot b$) and a closed 1-manifold $S \subset V \subset A \cup B \cup C$. We denote $A \cup B \cup C$ by T . Any additional intersection points arise as the zero locus of a normal perturbation to S within the $(d-1)$ -manifold sheet of T in which it *locally* lives.

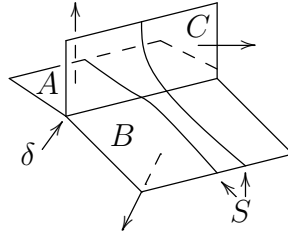


FIGURE 6.4.

But the total number of points $|\{q\}|$ along S where the normal field vanishes (as it reverses direction) must be even. If it were odd, S would reverse orientation on the R^{2k} normal bundle to its sheet in T . But this normal bundle is a direct sum of two R^k normal bundles: S within V and S within V' . S must either preserve or reverse the orientation of the normal R^k in these two bundles together. So the sum will certainly have its orientation preserved. Thus,

$$0 \equiv_2 x \cdot x = \left| \tilde{V} \cap \tilde{V} \right| \equiv_2 (a, Qb) + |\{q\}| \equiv_2 (a, Qb) + \text{even} = (a, Qb).$$

The first congruence is from the vanishing of $v_{k+1} \in H^{k+1}(X)$. \square

Remark on hypothesis of Lemma 6.4. As initially introduced, Figure 6.1 portrayed two loops l and α on the 2-sphere S^2 . But let us now reinterpret Figure 6.2 as two $(S^1 \times \mathbb{C}P^2)$ s, l and α , within $S^2 \times \mathbb{C}P^2$ by simply taking the Cartesian product of the entire diagram with $\mathbb{C}P^2$. Thus M is no longer four points but $\{4 \text{ points}\} \times \mathbb{C}P^2$. $X = S^2 \times \mathbb{C}P^2$ has a nontrivial, low dimensional, Stiefel-Whitney class: $w_2(\tau(X))$ is the generator of $H^2(S^2 \times \mathbb{C}P^2)$ corresponding to $H^2(* \times \mathbb{C}P^2)$ under the Künneth decomposition. So Lemma 6.4 does not apply, and indeed the sign formula in Theorem 6.1 is not valid in this example. Since W_l produces isotopic “before” and “after” submanifolds $S^1 \times \mathbb{C}P^2$, we expect the sign to be $+1$; however,

$$-1^{s(l)} i^{\chi(l \cap \alpha)} = -1^3 i^{12} = -1.$$

\square

Returning to the proof of Theorem 6.1, consider the Mayer-Vietoris sequences for $B \cup C$, the initial multi-balloon; $A \cup C$, the final multi-balloon; and $A \cup B$, the Wilson balloon. We use the abbreviations $a \cap c$, $b \cap c$, and $a \cap b$ for the dimensions of the intersected kernels $A \cap C$, $B \cap C$, and $A \cap B$, respectively. The sequences read in part:

$$(1) \quad D \rightarrow R^{a \cap c} \rightarrow H_k(A \cap C) \rightarrow H_k(A) \oplus H_k(C) \rightarrow H_k(A \cup C) \rightarrow H_{k-1}(A \cap C) \rightarrow \cdots,$$

$$(2) \quad 0 \rightarrow R^{b \cap c} \rightarrow H_k(B \cap C) \rightarrow H_k(B) \oplus H_k(C) \rightarrow H_k(B \cup C) \rightarrow H_{k-1}(B \cap C) \rightarrow \cdots,$$

$$(3) \quad 0 \rightarrow R^{a \cap b} \rightarrow H_k(A \cap B) \rightarrow H_k(A) \oplus H_k(B) \rightarrow H_k(A \cup B) \rightarrow H_{k-1}(A \cap B) \rightarrow \cdots$$

A basic fact is that the alternating sum of dimensions in any exact sequence with field coefficients is zero. We are only interested in collecting mod 2 information so we consider just the non-alternating sum of dimensions. let $\Delta = s(A \cup C) + s(B \cup C)$, the mod 2 “difference” of semi-characteristics. For notational convenience, we

extend the definition of the semi-characteristic to odd dimensional manifolds with boundary such as A , B , and C , writing $s(A) = \sum_{i=1}^k b_i(A) \pmod 2$, where b_i is again the Z_2 -Betti number. From sequences 1 and 2 we see:

$$\begin{aligned} s(A \cup C) &= a \cap c + \sum_{i=0}^k b_i(\delta) + \sum_{i=0}^k b_i(A) + \sum_{i=0}^k b_i(C), \text{ and} \\ s(B \cup C) &= b \cap c + \sum_{i=0}^k b_i(\delta) + \sum_{i=0}^k b_i(B) + \sum_{i=0}^k b_i(C), \text{ so} \\ (6.7) \quad \Delta &= a \cap c + b \cup c + s(A) + s(B), \text{ and by Lemma 6.3} = \frac{b_k(\delta)}{2} + a \cap b + s(A) + s(B). \end{aligned}$$

But from sequence 3,

$$(6.8) \quad s(A \cup B) = a \cap b + s(A) + s(B) + \sum_{i=1}^k b_i(M).$$

But

$$(6.9) \quad \sum_{i=1}^k b_i(M) \equiv \frac{\chi(M)}{2} + \frac{b_k(M)}{2} \pmod 2$$

so combining lines (6.7), (6.8), and (6.9) we get

$$(6.10) \quad \Delta = s(A \cup B) + \frac{\chi(M)}{2}$$

Thus the change Δ in the semi-characteristic of a fluctuating hypersurface $(A \cup C)$ upon NOT on the support $(A \cup B)$ of the Wilson balloon operator is given by the right-hand side of (6.10), a local formula in agreement with our sign rule. \square

Note 6.5. *The computation of $x \cdot x$ above relies on a certain regularity of the cellular k -chain V representing the class x . We have implicitly used that V is a pseudo-manifold, meaning that it is a union of $(k+1)$ -cells glued together so that every k -face is on the boundary of precisely two $(k+1)$ -cells.*

7. EXCITATIONS

Given a GDS on some manifold, and given a ground state Ψ , the state $O\Psi$ may be an excited state for certain choices of the operator O . We now briefly discuss these excitations.

One interesting choice of the operator O to consider is a product of Pauli Z over all $(d-1)$ -cells that intersect some arc on the dual lattice; we call these dual open Wilson loops. This case is completely analogous to the case of the GTC. This operator commutes with all terms of the Hamiltonian *except* those terms near the

two endpoints of the arc. The state $O\Psi$ is the same for any two arcs that are homologous. The excitations created near the endpoints of the arc must occur in pairs: given such a ground state Ψ and such an operator O , there is no state whose reduced density matrix agrees with Ψ everywhere except near *one* of the endpoints of the arc, while agreeing with the reduced density matrix of $O\Psi$ near that endpoint.

We can also consider operators obtained from “open balloons”. Assume $d = 2k + 2$, $k > 0$. Choose L to be an “open balloon” (a codimension = 1 submanifold with boundary). Let α be the $(d - 1)$ -dimensional cellular submanifold supporting a basis vector. In this case, choose the operator to be $\pm \text{NOT}$, where the sign is given by

$$(-1)^{s(L)} (i\chi(\partial(L \cap \alpha))) .$$

Note that in the DS model for $d = 2$, the operators that drag semions b and that drag semions \bar{b} are related by complex conjugation; however, for the present theory, if we replace i by $-i$ in the definition of the open balloon operator, then the resulting operator is unchanged because $\chi(\partial(L \cap \alpha))$ is even.

Balloon or open balloon operators will either commute or anti-commute with dual Wilson loop or dual open Wilson loop operators depending upon the parity of the intersection. Further, dual open Wilson loop operators commute with each other. The commutation relations between open balloons are more complicated, and we leave this as an open question. We also leave the statistics of the excitations created by the open balloons as an open question.

8. COMPARISON WITH Z_2 -DIJKGRAAF-WITTEN THEORIES

One aspect of the GDS theory (in $d = \text{even}$ dimensions) is that topological features of the d -manifold on which X it lives may create “missing sectors,” elements $\alpha \in H_{d-1}(X; Z_2)$ so that for hypersurface $E \subset X$, $[E] = \alpha$, $\psi[E] = 0$ for any zgs wf. An example of this is $\mathbb{C}P^{2k}$. Since $\chi(\mathbb{C}P^{2k}) = 2k + 1 = \text{odd}$ and $H_{4k-1}(\mathbb{C}P^{2k}; Z_2) = 0$, the Hilbert space $\text{GDS}(\mathbb{C}P^{2k}) = 0$.

It is natural to wonder if there is a relation between GDS and the twisted Z_2 -Dijkgraaf-Witten (DW) theory, and we thank Meng Cheng and Kevin Walker for raising this question and discussions on the resolution. For $G = Z_2$, $\text{BG} = K(Z_2, 1) = RP^\infty$, the infinite real projective space.

$$H^j(RP^\infty; R/Z) \cong \begin{cases} 0, & j \text{ even} \\ Z_2, & j \text{ odd} \end{cases} ,$$

so there is a unique twisting to consider when j is odd. Generically the effect on the Hilbert space of a twisting is to introduce some relations into the corresponding untwisted space, so we should check if these relations could cause the “missing sectors” observed in GDS. Theorem 8.1 below shows this is *not* the case.

In the untwisted DW theory for a finite group G , the Hilbert space $\text{DW}(X^d) =$ is \mathbb{C} -linear formal combinations of isomorphism classes of G -principal bundles over X . A twist is given by the choice of a cocycle α , $[\alpha] \in H^{d+1}(\text{BG}; R/Z)$. (The class $[\alpha]$ determines the theory.) Now, following [10], $\text{TDW}(X^d)$ is generated by maps $f : X \rightarrow \text{BG}$ with a relation for every homotopy $F : X \times I \rightarrow \text{BG}$, $F|_{X \times 0} = f_0$ and $F|_{X \times 1} = f_1$ of the form $f_0 = \omega f_1$, where $\omega = F^*(\alpha)[X \times I]$. From this definition, it follows easily that if for all maps $g : X \times S^1 \rightarrow \text{BG}$, $g^*(\alpha)[X \times S^1] = 1$, then $\dim \text{TDW}(X) = \dim \text{DW}(X)$, i.e., there are no additional relations. We use this criteria to prove:

Theorem 8.1. *For d even, GDS is distinct from $Z_2 \text{TDW}$ (and also $Z_2 \text{DW}$, the untwisted theory).*

Proof. For simplicity, begin with $d = 4k$, where GDS is distinguished from both twisted and untwisted Dijkgraaf-Witten theory by

$$\text{GDS}(\mathbb{C}P^{2k}) = 0$$

and

$$Z_2 \text{TDW}(\mathbb{C}P^{2k}) \cong Z_2 \text{DW}(\mathbb{C}P^{2k}) \cong \mathbb{C}.$$

The first equality results from $\chi(\mathbb{C}P^{2k})$, the Euler characteristic, being odd which implies that the empty hypersurface may be “swept over” $\mathbb{C}P^{2k}$ and back to itself with an odd number of Morse transitions, each of which counts for (-1) in the wave function.

Since $\pi_1(\mathbb{C}P^{2k}) \cong 0$, there is only the trivial Z_2 -principal bundle so

$$\text{DW}(\mathbb{C}P^{2k}) \cong \mathbb{C}.$$

We will prove that any map $g : \mathbb{C}P^{2k} \times S^1 \rightarrow RP^\infty$ homotopically factors through the second factor:

$$\begin{array}{ccc} g : \mathbb{C}P^{2k} \times S^1 & \xrightarrow{g} & RP^\infty \\ \downarrow \pi_2 & \nearrow g' & \\ S^1 & & \end{array} \quad (\text{commutes up to homotopy})$$

This immediately implies (by a Z_2 -Stokes theorem) that

$$g^*(\alpha)[\mathbb{C}P^{2k} \times S^1] = g'^*[S^1] = 1,$$

the second equality, since S^1 has no higher homology with any coefficient system.

Now we construct g' . Fix $\theta \in S^1$ and consider

$$\begin{array}{ccccc} & & & & S^\infty \\ & & & \nearrow l & \downarrow \times 2 \\ \mathbb{C}P^{2k} & \xrightarrow{\text{id} \times \theta} & \mathbb{C}P^{2k} \times S^1 & \xrightarrow{g} & RP^\infty \end{array}$$

The lift l exists since $\pi_1(\mathbb{C}P^{2k}) = 1$. l extends to a map over cone $(\mathbb{C}P^{2k})$ and this cone further projects back into RP^∞ .

$$\begin{array}{ccc} \text{cone}(\mathbb{C}P^{2k}) & \xrightarrow{\bar{l}} & S^\infty \\ \uparrow & \nearrow l & \downarrow \times 2 \\ \mathbb{C}P^{2k} & & RP^\infty \end{array}$$

Homotopically, this factors g through $g_1 : \Sigma(\mathbb{C}P^{2k}) \vee S^1 \rightarrow RP^\infty$, Σ denoting suspension. But $\Sigma(\mathbb{C}P^{2k})$ is also simply connected so g_1 extends over a cone on

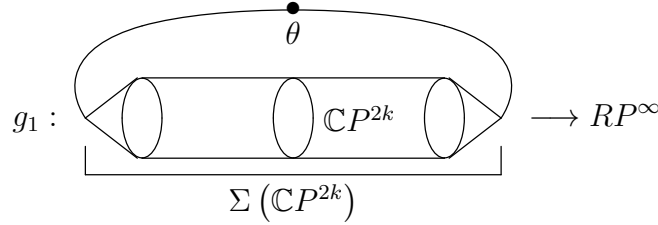


FIGURE 8.1.

$\Sigma(\mathbb{C}P^{2k})$, to a map g_2 ,

$$g_2 : \Sigma(\mathbb{C}P^{2k}) \vee S^1 \cup \text{cone}(\Sigma(\mathbb{C}P^{2k})) \rightarrow RP^\infty.$$

But the source of g_2 is homotopy equivalent to S^1 , and making this identification, g_2 becomes the desired g' .

With this warmup we can now handle the case $d = 4k + 2$, $k \geq 1$. This is done by considering the Hilbert spaces for $RP^2 \times \mathbb{C}P^{2k}$. The argument will be an iterative version of the previous. Since $\chi(RP^2 \times \mathbb{C}P^{2k}) = 2k + 1 = \text{odd}$, it again will suffice to show that for $g : RP^2 \times \mathbb{C}P^{2k} \times S^1 \rightarrow RP^\infty$ homotopically factors through a map $g' : RP^2 \times S^1 \rightarrow RP^\infty$ (since $H^{d+1}(RP^2 \times S^1; R/S^1) \cong 0$).

To simplify notation, set $M = RP^2 \times S^1$. Initially we have a (product) bundle:

$$\begin{array}{ccccc} \mathbb{C}P^{2k} & \longrightarrow & \mathbb{C}P^{2k} \times M & \xrightarrow{g} & RP^\infty \\ & & \downarrow & \nearrow h & \\ & & M & & \end{array}$$

where the homotopy extension h exists since $\pi_1(\mathbb{C}P^{2k}) = 1$.

Make h transverse to $RP^{\infty-1} \subset RP^\infty$ with $Y = h^{-1}(RP^{\infty-1})$, and write $M = Y \cup P$, $P = M \setminus Y$. By the previous method, g factors through g_1 ,

$$g_1 : [\nu_Y \times \mathbb{C}P^{2k} \cup (\partial(\nu_Y) \times \text{cone}(\mathbb{C}P^{2k}))] \cup P \rightarrow RP^\infty,$$

where ν_Y denotes the normal bundle to Y in M . Notice the space in brackets is a bundle E over Y with a simply connected fiber: $\Sigma(\mathbb{C}P^{2k})$:

$$\begin{array}{ccc} \Sigma(\mathbb{C}P^{2k}) & \longrightarrow & E \xrightarrow{g_1|} RP^\infty \\ & & \downarrow \quad \nearrow h_1 \\ & & Y \end{array}$$

As before, $\pi_1(\Sigma\mathbb{C}P^{2k}) = 1$ implies a homotopy factoring through h_1 . Now make h_1 transverse to $RP^{\infty-1} \subset RP^\infty$ and set $h_1^{-1}(RP^{\infty-1}) = L \subset Y$. Set $Y \setminus L = Q$, $Y = L \cup Q$.

Now g_1 factors through g_2 ,

$$g_2 : [(\nu_L \times \Sigma(\mathbb{C}P^{2k})) \cup \partial(\nu_L) \times \text{cone}(\Sigma(\mathbb{C}P^{2k}))] \cup Q \cup P \rightarrow RP^\infty.$$

Again the space in brackets is a bundle with simply connected fiber: $\Sigma(\Sigma(\mathbb{C}P^{2k}))$:

$$\begin{array}{ccc} \Sigma^2(\mathbb{C}P^{2k}) & \longrightarrow & F \xrightarrow{g_2|} RP^\infty \\ & & \downarrow \quad \nearrow h_2 \\ & & L \end{array}$$

Finally, make h_2 transverse to $RP^{\infty-1} \subset RP^\infty$ with $h_2^{-1}(RP^{\infty-1}) = J$ and $L \setminus J = K$, $L = J \cup K$. Now g_2 factors through g_3 ,

$$g_3 : [(\nu_J \times \Sigma^2(\mathbb{C}P^{2k})) \cup \partial(\nu_J) \times \text{cone}(\Sigma^2(\mathbb{C}P^{2k}))] \cup K \cup Q \cup P \rightarrow RP^\infty.$$

Again the space in brackets is a bundle with simply connected fiber $\Sigma^3(\mathbb{C}P^3)$:

$$\begin{array}{ccc} \Sigma^3(\mathbb{C}P^{2k}) & \longrightarrow & G \xrightarrow{g_3|} RP^\infty \\ & & \downarrow \\ & & J \end{array}$$

But now J is merely a finite number of points indexing the components of G . Each such component can now be lifted to S^∞ , coned in S^∞ , and then projected back to RP^∞ . The result is to factor g_3 through g_4 where the source of g_4 is the source of g_3 with these cones attached to the components of G . But with these cones the source of g_4 is homotopy equivalent to $RP^2 \times S^1 = J \cup K \cup Q \cup P$. Using this homotopy equivalence, g_4 may be identified with the map we have been seeking, g' . \square

9. DISCUSSION AND RELATION TO OTHER TQFTs

We have constructed a Hamiltonian which generalizes the double semion TQFT to higher dimensions and we have computed the dimension of the space of zero energy states for several different choices of ambient space. For odd dimensions

d , we have shown (Remark 5.7) that the ground state subspace of the theory is related that of the toric code by a local unitary transformation.

For even dimensions d , we conjecture that: for all non-empty choices of ambient space, there is no local unitary transformation that maps the ground state subspace of the GDS theory into the ground state subspace of the GTC or twisted Z_2 Dijkgraaf-Witten model. More strongly, we conjecture that: given a ball B contained in the ambient space, no ground state wavefunction of the GDS theory can be mapped by a local unitary transformation in such a way that the reduced density matrix on B agrees with the reduced density matrix on B of either a ground state wavefunction of the GTC or a ground state wavefunction of the twisted Z_2 Dijkgraaf-Witten model. To make this conjecture more precise, one may consider a sequence of cellulations of the ambient space with bounded local geometry and increasing number of cells so that the diameter of the cells tends to zero; then one constructs a sequence of quantum circuits which map some ground state wavefunction of the GDS theory so that the reduced density matrix on B has the desired property. These circuits are chosen to minimize the depth of the circuit (taking the gates in the circuit to have bounded range); then the conjecture is that this depth diverges.

This conjecture is one way of formally stating that the Hamiltonians correspond to different phases of matter. The main evidence for this belief in even dimensions is that we have shown that for certain choices of ambient space the ground state degeneracy of the GDS model does not agree with that of either the toric code or the twisted Z_2 Dijkgraaf-Witten model; this immediately implies that there is no local unitary transformation that maps the ground subspace of one theory onto that of the other theory in those cases as the dimensions are different. However, more work is needed to establish the stronger conjecture above. This is perhaps reminiscent of the case in two dimensions where it was shown first that no ground state of the toric code on a torus can be mapped by a local quantum circuit to a product state[11], then it was shown that the ground state of the toric code on a sphere could not be mapped to a product state[12], and finally more general invariants were found showing that the topological S-matrix of the theory is invariant under local unitary transformations[13]. Hence, to show such a conjecture here may require a more detailed analysis of excitations of the theory. This may require a description in terms of higher categories[14].

REFERENCES

- [1] R. Dijkgraaf and E. Witten, “Topological gauge theories and group cohomology”, *Comm. Math. Phys.* **129**, 393-429 (1990).
- [2] A. Kitaev, “Fault-tolerant quantum computation by anyons”, *Ann. Phys.* **303**, 2 (2003).
- [3] M. Freedman, D. Meyer, and F. Luo. “ Z_2 -systolic freedom and quantum codes”, in *Math. of Quantum Computation*, pages 287320, eds. R. K. Brylinski and G. Chen, CRC Press (Boca Raton) 2002.

- [4] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, “Topological quantum memory”, J. Math. Phys. **43**, 4452-4505 (2002).
- [5] M. A. Levin and X.-G. Wen, “String-net condensation: A physical mechanism for topological phases”, Phys.Rev. B **71**, 045110 (2005).
- [6] F. Quinn, “Lectures on Axiomatic Topological Quantum Field Theory”, in *Geometry and Quantum Field Theory*, page 323, eds. D. S. Freed and K. K. Uhlenbeck, IAS/Park City Mathematics, Vol 1, American Mathematical Society, IAS (Princeton, NJ) 1991.
- [7] K. Walker and Z. Wang, “(3+1)-TQFTs and Topological Insulators”, Front. Phys. **7**, 150 (2012), arXiv:1104.2632.
- [8] S. Bravyi, M. B. Hastings, and S. Michalakis, “Topological quantum order: stability under local perturbations”, J. Math. Phys. **51** 093512 (2010).
- [9] S. Bravyi and M. B. Hastings, “A short proof of stability of topological order under local perturbations”, Commun. Math. Phys. **307**, 609 (2011).
- [10] D. Freed and F. Quinn. *Chern-Simons Theory with Finite Gauge Group*, Comm. Math. Phys. 156 No. 3, 435—472 (1993).
- [11] S. Bravyi, M. B. Hastings, F. Verstraete, “Lieb-Robinson bounds and the generation of correlations and topological quantum order”, Phys. Rev. Lett. **97**, 050401 (2006).
- [12] M. B. Hastings, “Locality in Quantum Systems”, arXiv:1008.5137, Les Houches Lecture Notes, in *Quantum Theory from Small to Large Scales, Volume 95, 2010*, eds. J. Frohlich, M. Salmhofer, V. Mastropietro, W. De Roeck, and L. F. Cugliandolo, Oxford University Press (Oxford), 2012.
- [13] J. Haah, “An invariant of topologically ordered states under local unitary transformations”, arXiv:1407.2926.
- [14] L. Kong, X.-G. Wen, and H. Zheng, “Boundary-bulk relation for topological orders as the functor mapping higher categories to their centers”, arXiv:1502.01690.

APPENDIX A. GENERIC CELLULATIONS

Consider any set S of points in general position in d -dimensions. Here “general position” means that no more than $d+1$ points lie on any hypersphere and that no more than $n+1$ points lie on any n -dimensional hyperplane for $n < d$. Consider the Voronoi cells defined by this set of points. Consider any subset T of the d -dimensional Voronoi cells. Let B be the boundary of this subset. We now show that B is a $d-1$ -dimensional PL manifold.

Let x be any point in B . Then, x is at some distance D from some set of points p_1, p_2, \dots, p_k in which are centers of cells in T and points q_1, \dots, q_l which are centers of cells not in S . Thus, there are $k+l$ points in S at distance D from x . Because the points in S are in general position, $k+l \leq d+1$. Further, $k > 0$ and $l > 0$ (otherwise, x is not on the boundary), so $k, l \leq d$.

For notational simplicity, translate the point x to the origin so $|p_i| = |q_j| = D$. The points in B near the origin are the points that are the same distance from centers of cells in T as from centers of cells not in T . For any point y , the distance from y to a point p_i is equal to $\sqrt{d^2 - 2p_i \cdot y + y^2}$. So, a point y near the origin (i.e., sufficiently close to the origin that $p_1, \dots, p_k, q_1, \dots, q_l$ are closer to y than any

other point in S) is in B if and only if

$$(A.1) \quad \max_i p_i \cdot y = \max_j q_j \cdot y.$$

Since the points are in general position, the vectors p_i, q_j span at least $k + l - 1$ dimensions. Suppose that the vectors p_i, q_j span $k + l - 1$ dimensions. Then, the set of points satisfying the Eq. (A.1) is $R^{d+1-k-l}$ times the set of points in a $k + l - 1$ dimensional space (i.e., the space which is the span of p_i, q_j) which satisfy Eq. (A.1). So, in this case we can reduce the problem of showing that B is a PL manifold to the problem of showing that B is a PL manifold in this case that $k + l = d + 1$.

Suppose instead that the vectors v span $k + l$ dimensions. Then, there is some nonzero vector v such that $p_i \cdot v = q_j \cdot v = \text{constant}$ for all i, j . Then, translating the origin by this vector v , we arrive at a new problem in which the vectors p_i, q_j span only $k + l - 1$ dimensions. This reduces to the above case where p_i, q_j span $k + l - 1$ dimensions (this vector v is a vector in the space $R^{d+1-k-l}$ above). So, in general we can assume that $k + l = d + 1$ and the points span d dimensions.

Consider the space of vectors v such that

$$(A.2) \quad v \cdot p_i = v \cdot p_j$$

for all i, j and

$$(A.3) \quad v \cdot q_i = v \cdot q_j$$

for all i, j . These equations give $(k - 1) + (l - 1) = d - 1$ constraints. Hence, there is at least a one dimensional space of vectors satisfying Eqs. (A.2,A.3). Let v be a nonzero vector satisfying Eqs. (A.2,A.3) with $v \cdot p_i = z_p$ for some constant z_p and $v \cdot q_j = z_q$ for some constant z_q . By the assumption that the points are in general position, $z_p \neq z_q$, as otherwise there would be $d + 1$ points on the hyperplane of fixed inner product with v .

There is a $d - 1$ dimensional space of points orthogonal to v . Let m be any vector in this space. Then,

$$(A.4) \quad y = m + cv$$

satisfies Eq. (A.1) if and only if

$$(A.5) \quad c = \frac{\max_i p_i \cdot m - \max_j q_j \cdot m}{z_q - z_p}.$$

So, setting $y = m + \frac{\max_i p_i \cdot m - \max_j q_j \cdot m}{z_q - z_p} v$ gives a PL homeomorphism from a neighborhood of the origin in R^{d-1} to a neighborhood of the origin in B .

Note that without the assumption of general position, B need not be a manifold. As an example consider four points in the plane, at coordinates $(\pm 1, \pm 1)$. Consider the set of Voronoi cells containing the cells corresponding to the points at $(+1, -1)$

and $(-1, +1)$. Then, the boundary of this set is the horizontal and vertical axes, which has a singularity at the origin.